Optimal control of Volterra integral diffusions and application to contract theory

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October 29, 2024

Abstract

This paper focuses on the optimal control of a class of stochastic Volterra integral equations. Here the coefficients are regular and not assumed to be of convolution type. We show that, under mild regularity assumptions, these equations can be lifted in a Sobolev space, whose Hilbertian structure allows us to attack the problem through a dynamic programming approach. We are then able to use the theory of viscosity solutions on Hilbert spaces to characterise the value function of the control problem as the unique solution of a parabolic equation on Sobolev space. We provide applications and examples to illustrate the usefulness of our theory, in particular for a certain class of time-inconsistent Principal-Agent problems. As a by-product of our analysis, we introduce a new Markovian approximation for Volterra-type dynamics.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denote a fixed probability space, endowed with a standard $(\mathbb{F}, \mathbb{P})$–Brownian motion W of dimension $d \in \mathbb{N}^\star$. We consider the following type of controlled stochastic Volterra integral equation

$$X_t^\alpha = x + \int_0^t b_r(t, X_r^\alpha, \alpha_r)\mathrm{d}r + \int_0^t \sigma_r(t, X_r^\alpha, \alpha_r)\mathrm{d}W_r, \tag{1.1}$$

where $x \in \mathbb{R}^n$, α lives in an appropriate space of controls $\mathcal{A}$ taking their values in some Polish space A, and $b : [0,T]^2 \times \mathbb{R} \times A \longrightarrow \mathbb{R}^n$ and $\sigma : [0,T]^2 \times \mathbb{R} \times A \longrightarrow \mathcal{M}_{d,n}(\mathbb{R})$ are continuous in all their variables. We are interested in the optimal control problem

$$V_0(x) \coloneqq \sup_{\alpha \in \mathcal{A}} \mathbb{E}\bigg[\int_0^T f(t, X_t^\alpha, \alpha_t)\mathrm{d}t + g(X_T^\alpha)\bigg], \tag{1.2}$$

where $f : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are Borel-measurable functions.

Given their large scope of applications, stochastic control problems of the form (1.1)–(1.2) have received strong attention in the scientific literature. They have for example raised interest in medical sciences (see e.g. Schmiegel [35] and Saeedian, Khalighi, Azimi-Tafreshi, Jafari, and Ausloos [34]) or in finance, in particular in the study of rough volatility models (see e.g. Bayer, Friz, and Gatheral [6] or Gatheral, Jaisson, and Rosenbaum [18]).

More recently, Hernández and Possamaï showed in [24] that Volterra-type control problems naturally arise in contracting problem involving some form of time-inconsistency. These problems have quite specific features, and could be referred as *extended Volterra control problems*. More precisely, the state process in this class of problem is a uncountable family of processes $\mathbf{X}^\alpha \coloneqq \{X^{\alpha,s}\}_{\{s\in[0,T]\}}$, with

$$X_t^{\alpha,s} = x^s + \int_0^t b_r(X_r^{\alpha,r}, X_r^{\alpha,s}, \alpha_s)\mathrm{d}r + \int_0^t \sigma_r(X_r^{\alpha,r}, X_r^{\alpha,s}, \alpha_s)\mathrm{d}W_r. \tag{1.3}$$

Note in particular that, when b and σ do not depend on $X_r^{\alpha,r}$, the diagonal process $t \longmapsto X_t^{\alpha,t}$ satisfies the stochastic Volterra-integral equation (1.1).

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Volterra-type control problems have also attracted strong attention for their challenging mathematical features. Indeed, due to the presence of the t in b and σ, it is well known that the optimisation problem (1.2) is time inconsistent: X^α is not Markov or not even a semi-martingale in general, and therefore the flow property does not apply. Various techniques have been considered to overcome this difficulty. One rather popular method is to handle the problem through a maximum principle approach, see *e.g.* Agram and Oksendal [4], Agram, Oksendal, and Yakhlef [3], Lin and Yong [27] and Hamaguchi [21]. We also mention the recent contribution of Cárdenas, Pulido, and Serrano [11], who search for an optimal control in a relaxed form.

A large number of papers focus on recovering time-consistency by embedding the problem in a larger space, in which the new state process satisfies the usual flow property. Those are often referred as using the *lifting* approach. In the contributions of Abi Jaber, Miller, and Pham [2], di Nunno and Giordano [12] and Hamaguchi [20], the kernel of the stochastic Volterra integral equation writes as the linear transform of some element defined on a appropriate Banach space (and even in a Hilbert space in the case of [20]). This linear transformation involves some semi-group structure, so that it is possible to write the state process X^α as the image of an infinite dimensional process $\mathbf{X}^\alpha$ by the same transformation, with $\mathbf{X}^\alpha$ satisfying an infinite dimensional stochastic differential equation (SDE for short) and therefore satisfying the Markov property. In a slightly different approach, Viens and Zhang [38] lifts the state process—typically a fractional Brownian motion—in the Banach space of continuous path, treating the 'Volterra time' (the t in b and σ in (1.1)) as a parameter. This approach has been used in several subsequent works, such as the ones of Wang, Yong, and Zhang [39] and Wang, Yong, and Zhou [40].

We propose a new lifting in the same spirit as [38], in the case of regular kernels. More precisely, we also treat the 'Volterra time' as a parameter, thus lifting the state process in a space of paths. We shall however assume that the coefficients b and σ of the equation (1.3) are differentiable in this parameter, in the sense of Sobolev derivatives. This provides a Hilbertian structure as well as continuity in the parameter, which is crucial to connect the original problem (1.1)–(1.2) with the lifted one. We are then able to recover time-consistency for the lifted problem, and to study the control problem through a dynamic programming approach. In particular, we can resort to the standard theory of viscosity solutions on Hilbert spaces, see *e.g.* Lions [28; 29; 30]. Inaddition, and unlike the existing works in the literature, our approach enables us to study extended Volterra control problems (1.3) and to treat the control of the 'classical' Volterra dynamics (1.1) as a special case.

The paper is organised as follows. In Section 2, we introduce precisely the lifted space and process, and highlight their main properties. In Section 3, the value function of the infinite dimensional problem is characterised as the unique viscosity solution to a parabolic equation on Sobolev space. A special attention is also given to the case of uncontrolled volatility. We apply our theory to some examples in Section 4, with a particular emphasis on the contracting problem with sophisticated agent. Section 5 discusses an interest by-product of our analysis, which is a new Markovian representation of stochastic Volterra processes. Finally, Section 6 compares our contribution with some of the aforementioned references and Section 7 discusses the case of singular kernels.

2 The infinite dimensional problem

Our main requirements to define a 'good' lifting are the following

- the state space must be a Hilbert space, and the lifted state process must satisfy some flow property, as this will enable us to apply the standard theory of viscosity solutions on Hilbert space for a large class of stochastic control problems;
- if the original process writes as (1.1), it must write as a continuous function of the lifted process.

2.1 Choice of the state space

Let $\mathbb{L}^2([0,T],\mathrm{d}t)$ denote the equivalence class of square integrable functions $\varphi:[0,T]\longrightarrow\mathbb{R}^n$ and $C^1_c((0,T))$ the set of C^1 functions ψ on $(0,T)$ such that ψ and ψ' have compact support. Introduce the Sobolev space

$$W^{1,2}([0,T]) \coloneqq \bigg\{u\in\mathbb{L}^2([0,T],\mathrm{d}t) : \exists u'\in\mathbb{L}^2([0,T],\mathrm{d}t),\ \int_0^T u(t)\varphi'(t)\mathrm{d}t=\int_0^T u'(t)\varphi(t)\mathrm{d}t,\ \forall\varphi\in C^1_c((0,T))\bigg\},\tag{2.1}$$

as well as its scalar product $\langle u,v\rangle_{W^{1,2}}\coloneqq\int_0^T u(t)v(t)\mathrm{d}t+\int_0^T u'(t)v'(t)\mathrm{d}t$. Let H be the space defined by

$$H\coloneqq\big\{\mathbf{x}\coloneqq(\mathbf{x}_1,\dots,\mathbf{x}_n):\mathbf{x}_k\in W^{1,2}([0,T])\ \forall\ k\in\{1,\dots,n\}\big\},$$

endowed with the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle_H \coloneqq \sum_{k=1}^{n} \langle \mathbf{x}_k, \mathbf{y}_k \rangle_{W^{1,2}}, \ (x, y) \in H^2,$$

and the corresponding norm $\|\mathbf{x}\|_H \coloneqq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_H}$, $x \in H$. $(H, \|\cdot\|_H)$ is a Hilbert space, which will be fundamental for us when resorting to the theory of viscosity solutions; furthermore, we have the following important result, which is crucial for the well-posedness of our lifted problem. This uses the notation $\mathcal{C} \coloneqq C^0([0,T], \mathbb{R})$ for the space of continuous functions from $[0,T]$ to $\mathbb{R}$.

Lemma 2.1. *We have $H \subset \mathcal{C}$. Moreover, the injection is compact, that is to say that there exists a constant $C \geq 0$ such that*

$$\|\tilde{\mathbf{x}}\|_\infty \leq C \|\mathbf{x}\|_H, \ \forall \mathbf{x} \in H,$$

where $\tilde{\mathbf{x}}$ denotes the continuous representative of $\mathbf{x}$.

Proof. We prove the result in the case $n = 1$. The general result is recovered by reiterating the proof coordinate-wise. Each element $\mathbf{x} \in H$ has a continuous representative $\tilde{\mathbf{x}} \in \mathcal{C}$ (see *e.g.* Brézis [9, Theorem 8.2]). Moreover, for all $\mathbf{x} \in H$, we have for any $s \in [0,T]$

$$|\tilde{\mathbf{x}}^t| \leq |\tilde{\mathbf{x}}^t - \tilde{\mathbf{x}}^s| + |\tilde{\mathbf{x}}^s| \leq \int_0^T |\mathbf{x}'(t)| \mathrm{d}t + |\tilde{\mathbf{x}}^s|,$$

where we denote $\tilde{\mathbf{x}}^s \coloneqq \tilde{\mathbf{x}}(s)$ for all $s \in [0,T]$. Therefore, integrating with respect to s leads to

$$|\tilde{\mathbf{x}}^t| \leq \int_0^T |\mathbf{x}'(t)| \mathrm{d}r + \frac{1}{T} \int_0^T |\mathbf{x}(r)| \mathrm{d}r.$$

Thus, taking the supremum over t on the left-hand side and using Cauchy–Schwarz's inequality and standard concavity inequalities, we obtain

$$\|\tilde{\mathbf{x}}\|_\infty \leq C \|\mathbf{x}\|_H, \tag{2.2}$$

for some constant C depending only on T. ■

Remark 2.2. *As a consequence of the above estimate, the mapping*

$$\Phi : [0,T] \times H \ni (t, \mathbf{x}) \longmapsto \tilde{\mathbf{x}}^t \in \mathbb{R}^n,$$

is continuous and even Lipschitz-continuous in $\mathbf{x}$.

2.2 The infinite dimensional dynamics

Let $b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times A \longrightarrow \mathcal{M}_{d,n}(\mathbb{R})$, and introduce the H-valued mapping

$$B(t, \mathbf{x}, a)(\cdot) \coloneqq b_t(\cdot, \tilde{\mathbf{x}}^t, \tilde{\mathbf{x}}^\cdot, a),$$

as well as the $\mathcal{L}(\mathbb{R}^d, H)$-valued mapping

$$\Sigma(t, \mathbf{x}, a)(\cdot) \coloneqq \sigma_t(\cdot, \tilde{\mathbf{x}}^t, \tilde{\mathbf{x}}^\cdot, a),$$

for all $(t, \mathbf{x}, a) \in [0,T] \times H \times A$, with $\mathcal{L}(\mathbb{R}^d, H)$ being the set of linear functions from $\mathbb{R}^d$ to H.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a standard d-dimensional Brownian motion W, and denote by $\mathbb{F}^W$ its natural filtration. Denote by $\mathcal{A}^W$ the set of $\mathbb{F}^W$–progressively measurable processes taking their values in A such that the H-valued SDE

$$\mathbf{X}_t^\alpha = \mathbf{x}_0 + \int_0^t B(r, \mathbf{X}_r^\alpha, \alpha_r) \mathrm{d}r + \int_0^t \Sigma(r, \mathbf{X}_r^\alpha, \alpha_r) \mathrm{d}W_r, \ t \in [0,T]. \ \mathbb{P}--\text{a.s.}, \tag{2.3}$$

has a unique strong solution. Our first result provides concrete sufficient conditions to guarantee the existence and the uniqueness of a H-valued solution to (2.3), at least whenever the control α is constant. The point here is that standard Lipschitz-continuity may not be sufficient in general, unlike in the finite-dimensional case.

Proposition 2.3. *For $\phi \in \{b, \sigma\}$, assume that there exists two functions ϕ^1 and ϕ^2 such that*

$$\phi_t(s, x, y, a) = \phi_t^1(s, x, a) + \phi_t^2(s, a)y, \text{ for all } (t, s, x, y, a) \in [0, T]^2 \times \mathbb{R}^2 \times A.$$

Assume furthermore that

(i) ϕ^1 is continuous in all its variables, Lipschitz-continuous in x uniformly in (t, s, a), and admits a Sobolev derivative w.r.t. s which is also continuous in all its variables and Lipschitz-continuous in x uniformly in (t, s, a);

(ii) ϕ^2 is continuous and uniformly bounded in all its variables, and admits a Sobolev derivative w.r.t. s which is also continuous and uniformly bounded in all its variables.

Then (2.3) *has a unique solution in H.*

Proof. We check the assumptions of Gawarecki and Mandrekar [19, Theorem 3.3]. For simplicity, we assume without loss of generality that $n = 1$; the adaptation of the proof for the general multidimensional case is straightforward. We first verify that B and Σ take their values in H whenever $\mathbf{x} \in H$. Indeed, we have, for all $s \in [0, T]$,

$$B(t, \mathbf{x}, a) = b_t(s, \tilde{\mathbf{x}}^t, \tilde{\mathbf{x}}^s, a) = b_t^1(s, \tilde{\mathbf{x}}^t, a) + b_t^2(s, a)\tilde{\mathbf{x}}^s.$$

By the assumptions made above and the fact that $\mathbf{x} \in H$, it is clear that $s \longmapsto b_t^1(s, \tilde{\mathbf{x}}^t, a)$ and $s \longmapsto b_t^2(s, a)\tilde{\mathbf{x}}^s$ admit Sobolev derivatives. Therefore, B takes its values in H. We handle the case of Σ in the exact same way.

We secondly verify that B and Σ have linear growth in $\mathbf{x}$, uniformly in (t, a). We denote by ∂_s the derivation w.r.t. s in the Sobolev sense. We have

$$\begin{aligned}
\|B(t, \mathbf{x}, a)\|_H^2 &= \int_0^T \big(b_t(s, \tilde{\mathbf{x}}^t, \tilde{\mathbf{x}}^s, a)\big)^2 \mathrm{d}s + \int_0^T \big(\partial_s (b_t(s, \tilde{\mathbf{x}}^t, \tilde{\mathbf{x}}^s, a))\big)^2 \mathrm{d}s \\
&= \int_0^T \big(b_t^1(s, \tilde{\mathbf{x}}^t, a) + b_t^2(s, a)\tilde{\mathbf{x}}^s\big)^2 \mathrm{d}s + \int_0^T \big(\partial_s b_t^1(s, \tilde{\mathbf{x}}^t, a) + \partial_s b_t^2(s, a)\tilde{\mathbf{x}}^s + b_t^2(s, a)(\widetilde{\partial_s \mathbf{x}})^s\big)^2 \mathrm{d}s.
\end{aligned}$$

Since b^1 and $\partial_s b^1$ are Lipschitz-continuous in their space variable, uniformly in (t, s, a), we have for some constant C

$$|b_t^1(s, \tilde{\mathbf{x}}^t, a)| + |\partial_s b_t^1(s, \tilde{\mathbf{x}}^t, a)| \le C(1 + |\tilde{\mathbf{x}}^t|) \le C(1 + \|\mathbf{x}\|_H),$$

see (2.2) for the latter inequality. Then, we easily deduce from the boundedness of b_t^2 and $\partial_s b_t^2$ that B, and similarly Σ, have quadratic growth in $\mathbf{x}$.

We finally prove that B and Σ are Lipschitz-continuous in $\mathbf{x}$. Fix $(\mathbf{x}, \mathbf{y}) \in H^2$, we have

$$\begin{aligned}
\|B(t, \mathbf{x}, a) - B(t, \mathbf{y}, a)\|_H^2 \le& \int_0^T \big(|b_t^1(s, \tilde{\mathbf{x}}^t, a) - b_t^1(s, \tilde{\mathbf{y}}^t, a)| + |b_t^2(s, a)\tilde{\mathbf{x}}^s - b_t^2(s, a)\tilde{\mathbf{y}}^s|\big)^2 \mathrm{d}s \\
&+ \int_0^T \big(|\partial_s b_t^1(s, \tilde{\mathbf{x}}^t, a) - \partial_s b_t^1(s, \tilde{\mathbf{x}}^t, a)| + |\partial_s b_t^2(s, a)\tilde{\mathbf{x}}^s - \partial_s b_t^2(s, a)\tilde{\mathbf{x}}^s| \\
&\quad + |b_t^2(s, a)\widetilde{\partial_s \mathbf{x}}^s - b_t^2(s, a)\widetilde{\partial_s \mathbf{y}}^s|\big)^2 \mathrm{d}s \\
\le& C\bigg(\int_0^T \big(|\tilde{\mathbf{x}}^t - \tilde{\mathbf{y}}^t| + |\tilde{\mathbf{x}}^s - \tilde{\mathbf{y}}^s|\big)^2 \mathrm{d}s + \int_0^T \big(|\tilde{\mathbf{x}}^s - \tilde{\mathbf{y}}^s| + |\widetilde{\partial_s \mathbf{x}}^s - \widetilde{\partial_s \mathbf{y}}^s|\big)^2 \mathrm{d}s\bigg),
\end{aligned}$$

for some constant $C > 0$, where we used the Lipschitz-continuity of b^1 and $\partial_s b^1$ and the boundedness of b^2 and $\partial_s b^2$. Recalling (2.2), we have

$$|\tilde{\mathbf{x}}^t - \tilde{\mathbf{y}}^t| \le \|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty \le \|\mathbf{x}^t - \mathbf{y}^t\|_H,$$

from which we finally deduce that B is Lipschitz-continuous in $\mathbf{x}$, uniformly in (t, a). We proceed similarly for Σ. ∎

Remark 2.4. *The linear dependence on $\tilde{\mathbf{x}}^s$ is required to obtain the Lipschitz-continuity of the coefficients. Without this, we could still obtain existence of weak solutions (in the sense of martingale problems), see* [19, §3.9]*. Note that when $\mathbf{X}$ is the lifted version of* Equation (1.1)*, the term in $\mathbf{x}^s$ is not involved in* (2.3)*, and therefore the well-posedness of* (2.3) *directly proceeds from the well-posedness of* (1.1)*.*

Remark 2.5. *This infinite dimensional process enjoys two important properties regarding the original Volterra-type dynamics* (2.3)

(*i*) *the process* $\mathbf{X}^\alpha$ *is solution to a stochastic differential equation, whereas the process* X^α *defined in* (1.1) *is solution to a stochastic integral equation; in particular,* $\mathbf{X}^\alpha$ *is a semi-martingale*;

(*ii*) *for* $x \in \mathbb{R}^n$, *denote* $\mathbf{p}(x)$ *the element of* H *such that* $\widetilde{\mathbf{p}}(x)^t = x$ *for all* $t \in [0,T]$. *Assume* $\mathbf{X}_0^\alpha = \mathbf{p}(x)$, *and that* b *and* σ *write as in* Equation (1.1). *Then, the diagonal process* $t \longmapsto X_t^{\alpha,t}$ *satisfies* SDE (1.1). *Therefore, if uniqueness holds for the latter, the diagonal process is equal to the original controlled dynamics,* $\mathbb{P}$–a.s. *The question of the existence and uniqueness of solutions of* (1.1) *has been studied for example by* Ito [26] (*see also Protter [33] and Pardoux and Protter [32]*), *and is mostly obtained by assuming that* b *and* σ *are Lipschitz-continuous in* x *and in the* '*Volterra time*' t.

2.3 The infinite dimensional control problem

From now, for $\mathbf{x} \in H$, we shall abuse the notation and still denote by $\mathbf{x}$ its continuous representative. Given the dynamics (2.3), we consider the control problem

$$\mathcal{V}_0(\mathbf{x}_0) \coloneqq \sup_{\alpha \in \mathcal{A}^W} \mathbb{E}\bigg[\int_0^T F(r, \mathbf{X}_r^\alpha, \alpha_r)\mathrm{d}r + G(\mathbf{X}_T^\alpha)\bigg], \tag{2.4}$$

for some $F : [0,T] \times H \times A \longrightarrow \mathbb{R}$ and $G : H \longrightarrow \mathbb{R}$. The following result states that this control problem is connected to the control problem of a Volterra-type SDE (1.2) in the following way.

Proposition 2.6. *Let* $x \in \mathbb{R}$. *Assume* b *and* σ *write as in* Equation (1.1), *and that* $F(t, \mathbf{x}, a) = f(t, x^t, a)$ *and* $G(\mathbf{x}) = g(x^T)$ *for all* $(t, \mathbf{x}, a) \in [0,T] \times H \times A$, *with* f *and* g *also from* Equation (1.1). *Assume furthermore that uniqueness holds for* (1.1). *Then* $\mathcal{V}_0(\mathbf{p}(x)) = V_0(x)$.

Proof. As pointed in Remark 2.5, when the initial condition of the lifted process $\mathbf{X}^\alpha$ is $\mathbf{p}(x)$, and when uniqueness holds for Equation (1.1), the diagonal process pf $\mathbf{X}^\alpha$ and the original Volterra-type dynamics X^α are equal $\mathbb{P}$–a.s. By definition of F and G, and since we optimise on the same set $\mathcal{A}$ of open-loop controls, the two value functions are the equal. ∎

Remark 2.7 (Choice of the set of controls)**.** *The choice of the set of controls is crucial for the above proposition. Indeed, if we choose to consider closed-loop controls for either* (1.2) *or* (2.4), *then we might have* $\mathcal{V}_0(\mathbf{p}(x)) \neq V_0(x)$, *as the filtration generated by* X, *the filtration generated by* $\mathbf{X}$ *and* $\mathbb{F}^W$ *are different in general. However, if we consider the infinite dimensional control problem* (2.4) *as an object on its own—for example motivated by the study of moral hazard questions for time-inconsistent agents, see* [24]*—then we may either consider closed-loop or open-loop controls: both cases can be encapsulated in our dynamic programming approach.*

3 Dynamic programming equation

3.1 The value function

We introduce a dynamic version of the control problem (2.4). Denoting by $\mathbf{X}^{t,\mathbf{x},\alpha}$ the solution of (2.3) such that $\mathbf{X}_t^{t,\mathbf{x},\alpha} = \mathbf{x}$, we define:

$$\mathcal{V}(t, \mathbf{x}) \coloneqq \sup_{\alpha \in \mathcal{A}} \mathbb{E}\bigg[\int_0^T F(r, \mathbf{X}_r^{t,\mathbf{x},\alpha}, \alpha_r)\mathrm{d}r + G(\mathbf{X}_T^{\alpha,t,\mathbf{x}})\bigg], \text{ for all } (t, \mathbf{x}) \in [0,T] \times H. \tag{3.1}$$

Proposition 3.1 (Regularity of the value function)**.** *Assume that*

(*i*) F *is uniformly continuous in* $(t, \mathbf{x}) \in [0,T] \times B_H(0,R)$, *uniformly in* $a \in A$, *for all* $R \geq 0$, *where* $B_H(0,R)$ *denotes the ball of radius* R *and centre* 0 *for the metric* $\|\cdot\|_H$;

(*ii*) G *is uniformly continuous in* $\mathbf{x} \in B_H(0,R)$ *for all* $R \geq 0$;

(*iii*) F *and* G *have polynomial growth in* $\mathbf{x}$, *uniformly in the other variables.*

Then the value function $\mathcal{V}$ *is uniformly continuous on all the sets* $[0,T] \times B_H(0,R)$, $R \geq 0$, *and has polynomial growth in* $\mathbf{x}$ *uniformly in* t.

Proof. This a direct application of [16, Proposition 3.61]. ∎

Our lifted control problem (2.4) falls under the scope of Markovian control problems on Hilbert spaces, and we may therefore naturally formulate the following dynamic programming principle.

Proposition 3.2 (Dynamic programming principle)**.** *Under the assumptions of* Proposition 3.1*, we have*

$$\mathcal{V}(t,\mathbf{x}) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\bigg[\int_t^\theta F\big(r,\mathbf{X}_r^{t,\mathbf{x},\alpha},\alpha_r\big)\mathrm{d}r + \mathcal{V}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})\bigg],\ \forall (t,x)\in[0,T]\times H,\ \text{and}\ \theta\in\mathcal{T}_{t,T}, \tag{3.2}$$

where $\mathcal{T}_{t,T}$ denotes the set of $[t,T]$-valued $\mathbb{F}$–stopping times.

Proof. This is a direct application of [16, Proposition 2.24]. ∎

3.2 Viscosity solutions

For any smooth $\varphi : [0,T]\times H \longmapsto \mathbb{R}$, we denote by $\partial_t\varphi$ the derivative of φ with respect to $t\in[0,T]$, and by $D_{\mathbf{x}}\varphi$ and $D^2_{\mathbf{xx}}\varphi$ the first- and second-order Fréchet derivatives of φ with respect to $\mathbf{x}\in H$. For all $(t,\mathbf{x})\in[0,T]\times H$, by Riesz's representation theorem, $D_{\mathbf{x}}\varphi(t,\mathbf{x})$ can be identified to an element of H, and $D^2_{\mathbf{xx}}\varphi(t,\mathbf{x})$ to an endomorphism of H.

The purpose of this section is to show that the value function $\mathcal{V}$ of (2.4) can be characterised as the unique viscosity solution of the dynamic programming equation

$$-\partial_t u(t,\mathbf{x}) - \sup_{a\in A}\bigg\{\big\langle D_{\mathbf{x}}u(t,\mathbf{x}), b_t(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a)\big\rangle_H + \frac{1}{2}\big\langle\sigma_t(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a), D^2_{\mathbf{xx}}u(t,\mathbf{x})\sigma(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a)\big\rangle_H + F(t,\mathbf{x},a)\bigg\} = 0, \tag{3.3}$$

with terminal condition $u|_{t=T} = G$.

Definition 3.3 (Viscosity solutions)**.** *Let $u : [0,T]\times H \longrightarrow \mathbb{R}$ be locally bounded.*

(i) u is said to be a viscosity super-solution of (3.3) *if $u(T,\cdot)\geq G$ and, for all $\varphi\in C^{1,2}([0,T]\times H)$ such that $u-\varphi$ has a local minimum in $(t,\mathbf{x})$, we have*

$$-\partial_t\varphi(t,\mathbf{x}) - \sup_{a\in A}\bigg\{\big\langle D_{\mathbf{x}}\varphi(t,\mathbf{x}), b_t(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a)\big\rangle_H + \frac{1}{2}\big\langle\sigma_t(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a), D^2_{\mathbf{xx}}\varphi(t,\mathbf{x})\sigma(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a)\big\rangle_H + F(t,\mathbf{x},a)\bigg\} \geq 0.$$

(ii) u is said to be a viscosity sub-solution of (3.3) *if $u(T,\cdot)\leq G$ and, for all $\varphi\in C^{1,2}([0,T]\times H)$ such that $u-\varphi$ has a local maximum in $(t,\mathbf{x})$, we have*

$$-\partial_t\varphi(t,\mathbf{x}) - \sup_{a\in A}\bigg\{\big\langle D_{\mathbf{x}}\varphi(t,\mathbf{x}), b_t(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a)\big\rangle_H + \frac{1}{2}\big\langle\sigma_t(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a), D^2_{\mathbf{xx}}\varphi(t,\mathbf{x})\sigma(\cdot,\tilde{\mathbf{x}}^t,\mathbf{x}^{\cdot},a)\big\rangle_H + F(t,\mathbf{x},a)\bigg\} \leq 0.$$

(iii) u is said to be a viscosity solution of (3.3) *if it is both a viscosity super-solution and viscosity sub-solution of* (3.3)*.*

Then, applying standard viscosity theory on Hilbert space (see e.g. [16, Theorem 3.67]), we may formulate the following characterisation of $\mathcal{V}$.

Proposition 3.4. *Assume that*

(i) $t\longmapsto B_t(\mathbf{x},a)$ is continuous, uniformly in $(\mathbf{x},a)\in B_H(0,R)\times A$ for all $R>0$;

(ii) σ has linear growth in $\tilde{x}^t$ and $\tilde{x}^s$, uniformly in the other variables.

Then $\mathcal{V}$ is the unique continuous viscosity solution of (3.3) *with polynomial growth.*

Proof. Let $(e_k)_{k\in\mathbb{N}^\star}$ be an orthonormal basis of H. We essentially have to check [16, Assumption (3.155)], that is

$$\lim_{N\to\infty}\sup_{a\in A}\Big\{\mathrm{Tr}\big[\Sigma_t(\mathbf{x},a)\Sigma_t(\mathbf{x},a)^\top\mathcal{Q}_N\big]\Big\} = 0,\ \forall (t,\mathbf{x})\in[0,T]\times H, \tag{3.4}$$

where $\mathcal{Q}_N$ is the orthonormal projection onto the family $(e_k)_{k\in\mathbb{N}^\star\setminus\{1,\dots,N\}}$. Note that $\Sigma_t(\mathbf{x},a)\Sigma_t(\mathbf{x},a)^\top$ corresponds to the endomorphism of H

$$\mathbf{y}\longmapsto\langle\sigma_t(\cdot,\tilde{x}^t,\tilde{x}^{\cdot},a),\mathbf{y}\rangle_H\,\sigma_t(\cdot,\tilde{x}^t,\tilde{x}^{\cdot},a).$$

Then, denoting by $\sigma^k_t(\mathbf{x},a)$ the projection of $\Sigma_t(\mathbf{x},a)$ onto e_k, for $k\in\mathbb{N}^\star$, we have

$$\mathrm{Tr}\big[\Sigma_t(\mathbf{x},a)\Sigma_t(\mathbf{x},a)^\top\mathcal{Q}_N\big]=\sum_{k=N+1}^{\infty}|\sigma^k_t(\mathbf{x},a)|^2.$$

However, we have

$$|\sigma^k_t(\mathbf{x},a)|=|\langle\sigma_t(\cdot,\tilde{x}^t,\tilde{x}^{\cdot},a),e_k\rangle_H|\leq C(1+|\tilde{x}^t|)|\langle 1,e_k\rangle_H|+|\langle\mathbf{x},e_k\rangle_H|,$$

and therefore

$$\mathrm{Tr}\big[\Sigma_t(\mathbf{x},a)\Sigma_t(\mathbf{x},a)^\top\mathcal{Q}_N\big]\leq 2(1+|\tilde{x}^t|)^2\sum_{k=N+1}^{\infty}|\langle 1,e_k\rangle_H|^2+2\sum_{k=N+1}^{\infty}|\langle\mathbf{x},e_k\rangle_H|^2.$$

Since both 1 (as a constant mapping) and $\mathbf{x}$ belong to H, the two sums on the right-hand side go to 0 as $N\longrightarrow\infty$. Since this term is independent from a, we deduce that (3.4) holds true, and we may therefore conclude by applying [16, Theorem 3.67]. ■

3.3 The case of uncontrolled volatility

In this paragraph, we assume that σ does not depend on a, and that there exists a bounded $\theta:[0,T]\times H\times A\longrightarrow\mathbb{R}$ such that $B_t(\mathbf{x},a)=\Gamma_t(\mathbf{x})+\Sigma_t(\mathbf{x})\theta_t(\mathbf{x},a)$. We show that the value function of the infinite dimensional control problem can be expressed as the solution of a backward SDE. To this end, we reformulate the lifted control problem in weak formulation. Let $\mathbf{X}$ be the unique strong solution of the H-valued SDE

$$\mathbf{X}_t=\mathbf{x}_0+\int_0^t\Gamma_r(\mathbf{X}_r)\mathrm{d}r+\int_0^t\Sigma_r(\mathbf{X}_r)\mathrm{d}W_r,\ \mathbb{P}\text{–a.s.}$$

Let $\alpha\in\mathcal{A}$. By the existence of the function θ introduced above, it follows from Girsanov's theorem that there exists a probability measure $\mathbb{P}^\alpha$ equivalent to $\mathbb{P}$ such that

$$W^\alpha_t:=W_t-\int_0^t\theta_r(\mathbf{X}_r,\alpha_r)\mathrm{d}r,$$

is a $\mathbb{P}^\alpha$–Brownian motion. Therefore

$$\mathbf{X}_t=\mathbf{x}_0+\int_0^t B_r(\mathbf{X}_r,\alpha_r)dr+\int_0^t\Sigma_r(\mathbf{X}_r)dW^\alpha_r,\ \mathbb{P}^\alpha\text{–a.s.},$$

and we may reformulate the control problem in the following way

$$\mathcal{V}^w_0(\mathbf{x}_0)=\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\int_0^T F_r(\mathbf{X}_r,\alpha_r)dr+G(\mathbf{X}_r)\bigg].\tag{3.5}$$

We can easily see that an analogue of Proposition 2.6 holds true here; indeed, if X writes

$$X_t=x_0+\int_0^t\gamma_r(t,X_r)\mathrm{d}r+\int_0^t\sigma_r(t,X_t)\mathrm{d}W_r,\ \mathbb{P}\text{–a.s.},$$

and

$$V_0(x_0):=\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\mathbb{P}^\alpha}\bigg[\int_0^T f_r(X_r,\alpha_r)\mathrm{d}r+g(X_T)\bigg],$$

then $V_0(x_0)=\mathcal{V}_0(\mathbf{x}_0)$ whenever $\mathbf{x}_0=\mathbf{p}(x_0)$.

Let now $\mathcal{H}:[0,T]\times H\times\mathbb{R}\longrightarrow\mathbb{R}$ be the Hamiltonian defined by $\mathcal{H}_t(\mathbf{x},z)=\sup_{a\in A}\{z\theta_t(\mathbf{x},a)+F_t(\mathbf{x},a)\}$.

Proposition 3.5. *Assume that*

- $\mathcal{H}$ *is Lipschitz-continuous in* z;
- F *has linear growth in* $\mathbf{x}\in H$*, uniformly in the other variables*;
- Γ *and* Σ *are Lipschitz-continuous in* $\mathbf{x}\in H$*, uniformly in* $t\in[0,T]$.

Then $\mathcal{V}_0 = Y_0$*, where* (Y, Z) *is the unique solution of the backward* SDE

$$Y_t = G(\mathbf{X}_T) + \int_t^T \mathcal{H}_r(\mathbf{X}_r, Z_r)\mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r.$$

Furthermore, if there exists a measurable mapping $\psi : [0,T] \times H \times \mathbb{R} \longrightarrow A$ *such that*

$$\mathcal{H}(t, \mathbf{x}, z) = z\theta_t\big(\mathbf{x}, \psi_t(\mathbf{x}, z)\big) + F_t\big(\mathbf{x}, \psi_t(\mathbf{x}, z)\big), \tag{3.6}$$

then $\alpha_t^* \coloneqq \psi_t(\mathbf{X}_t, Z_t)$ *is an optimal control for* (3.5)*.*

Proof. For $\alpha \in \mathcal{A}$, let (Y^α, Z^α) denote the solution of the backward SDE:

$$Y_t^\alpha = G(\mathbf{X}_T) + \int_t^T \big(F_r(\mathbf{X}_r, \alpha_r) + Z_r^\alpha \theta_r(\mathbf{X}_r, \alpha_r)\big)\mathrm{d}r - \int_t^T Z_r^\alpha \mathrm{d}W_r.$$

By the Lipschitz and linear growth assumptions made on $\mathcal{H}$, Γ, Σ and F and the boundedness of θ, there exists a unique solution (Y^α, Z^α) to be above equation (see e.g. Pardoux and Peng [31], El Karoui, Peng, and Quenez [15] or Zhang [41]). As the equations solved by (Y, Z) and (Y^α, Z^α) satisfy the usual Lipschitz and measurability conditions, and by definition of $\mathcal{H}$, the comparison principle for backward SDEs ensures that $Y_0^\alpha \leq Y_0$. Since α is arbitrary, this shows that $\mathcal{V}_0 \leq Y_0$.

Fix now $\varepsilon > 0$. By measurable selection, there exists a measurable mapping $\psi^\varepsilon : [0,T] \times H \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\mathcal{H}_t(\mathbf{x}, z) \leq F_t(\mathbf{x}, \psi_t^\varepsilon(\mathbf{x}, z)) + z\theta_t(\mathbf{x}, \psi_t^\varepsilon(\mathbf{x}, z)) + \varepsilon.$$

Introducing $\alpha_t^\varepsilon \coloneqq \psi_t^\varepsilon(\mathbf{X}_t, Z_t)$, we have

$$\begin{aligned}
Y_t - Y_t^\varepsilon &= \int_t^T \big(\mathcal{H}_r(\mathbf{X}_r, Z_r) - F_r(\mathbf{X}_r, \alpha_r^\varepsilon) - Z_r^{\alpha^\varepsilon}\theta_r(\mathbf{X}_r, \alpha_r^\varepsilon)\big)\mathrm{d}r - \int_t^T (Z_r - Z_r^{\alpha^\varepsilon})\mathrm{d}W_r \\
&= \int_t^T \Big(\big(\mathcal{H}_r(\mathbf{X}_r, Z_r) - F_r(\mathbf{X}_r, \alpha_r^\varepsilon) - Z_r\theta_r(\mathbf{X}_r, \alpha_r^\varepsilon)\big) + (Z_r - Z_r^{\alpha^\varepsilon})\theta_r(\mathbf{X}_r, \alpha_r^\varepsilon)\Big)\mathrm{d}r - \int_t^T (Z_r - Z_r^{\alpha^\varepsilon})\mathrm{d}W_r \\
&= \int_t^T \big(\mathcal{H}_r(\mathbf{X}_r, Z_r) - F_r(\mathbf{X}_r, \alpha_r^\varepsilon) - Z_r\theta_r(\mathbf{X}_r, \alpha_r^\varepsilon)\big)\mathrm{d}r - \int_t^T (Z_r - Z_r^{\alpha^\varepsilon})\mathrm{d}W_r^{\alpha^\varepsilon} \\
&\leq \varepsilon(T - t) - \int_t^T (Z_r - Z_r^{\alpha^\varepsilon})\mathrm{d}W_r^{\alpha^\varepsilon}.
\end{aligned}$$

Thus, we have $Y_0 \leq Y_0^{\alpha^\varepsilon} + T\varepsilon$. By arbitrariness of ε, this implies that $Y_0 \leq \mathcal{V}_0$, and therefore the desired equality holds true. In particular, when (3.6) holds, we have $\mathcal{V}_0^w = Y_0 = Y_0^{\alpha^\star}$, which means that $\alpha^\star$ is an optimal control. ■

Remark 3.6. *Let us discuss what the assumption* $B_t(\mathbf{x}, a) = \Gamma_t(\mathbf{x}) + \Sigma_t(\mathbf{x})\theta_t(\mathbf{x}, a)$ *means in the context of the control of stochastic Volterra integral equations. If one wants to be able to apply Girsanov's theorem, the real-valued mapping* θ *must only depends on* $\mathbf{x}$ *and the 'regular time'* t*, and not on the 'Volterra time'* s*. This means that the dependency on* s *must be the same* b *and* σ*. This is for instance the case for the following dynamics, considered by* di Nunno and Giordano [12]

$$X_t = x + \int_0^t K(t - r)\Big(\big(b_r^1(X_r) + \sigma_r(X_r)b_r^2(X_r, \alpha_r)\big)\mathrm{d}r + \sigma_r(X_r)\mathrm{d}W_r\Big). \tag{3.7}$$

Therefore, this approach cannot be applied when the volatility is not Volterra, as it would enforces the drift to be standard as well.

4 Examples

4.1 A (very) simple starter

We start with a very simple example to familiarise the reader with our methodology. Consider the uncontrolled non-Volterra SDE:

$$\mathrm{d}X_t = X_t\mathrm{d}t + \mathrm{d}W_t.$$

The corresponding lifted dynamics (starting at time t at some $\mathbf{x} \in H$) is given by the flow $\mathbf{X}^{t,\mathbf{x}} := (X^{t,\mathbf{x},s})_{0\leq s\leq T}$ such that

$$X_r^{t,\mathbf{x},s} = \mathbf{x}^s + \int_t^r X_\theta^{t,\mathbf{x},\theta}\mathrm{d}\theta + W_\theta - W_t,\ \text{for all } s \in [0,T].$$

Our goal is to show that the function u defined by

$$u(t,\mathbf{x}) := \mathbb{E}\big[X_T^{t,\mathbf{x},T}\big],$$

satisfies the dynamic programming equation (3.3). For a fixed t, we first introduce the function $y(s) := \int_t^s \mathbb{E}\big[X_\theta^{t,\mathbf{x},\theta}\big]\mathrm{d}\theta$, for $s \in [t,T]$. We easily see that y satisfies the ODE

$$y'(s) = x^s + y(s),\ \text{with } y(t) = 0,$$

from which we deduce that $y(s) = \mathrm{e}^s \int_t^s x^r \mathrm{e}^r \mathrm{d}r$, and finally that

$$u(t,\mathbf{x}) = y(T) = x^T + \mathrm{e}^T \int_t^T x^s \mathrm{e}^{-s}\mathrm{d}s.$$

Note that this can be rewritten

$$u(t,\mathbf{x}) = \int_0^T x^s \mathrm{e}^{-s}\mathbf{1}_{[t,T]}(s)\mathrm{d}s. \tag{4.1}$$

We then compute $\partial_t u(t,\mathbf{x}) = -x^t\mathrm{e}^{T-t}$, and, observing that the Fréchet derivative of u with respect to $\mathbf{x}$ can be identified to the càdlàg function

$$D_\mathbf{x} u(t,\mathbf{x})(s) = \mathrm{e}^{T-s}\mathbf{1}_{[t,T]}(s).$$

We emphasise that this is simply a convenient representation of the differential of u, which is strictly speaking a linear form on H. According to Riesz's representation theorem, we could also write $D_\mathbf{x}(t,\mathbf{x})$ as an element of H. However, this would not be of any use in the present context where u is smooth. We recall that the lifting on the Sobolev space is essentially a way to apply the standard infinite dimensional viscosity solutions theory to our problem, which has no reason to be involved when we look for smooth value functions.

Next we compute

$$\begin{aligned}\partial_t u(t,\mathbf{x}) + \int_0^T D_\mathbf{x} u(t,\mathbf{x})(s) x^t \mu(\mathrm{d}s) &= -x^t\mathrm{e}^{T-t} + \int_t^T \mathrm{e}^{T-s}\mu(\mathrm{d}s) \\ &= -x^t\mathrm{e}^{T-t} + \int_t^T \mathrm{e}^{T-s}\mathrm{d}s + 1 = -x^t e^{T-t} + (\mathrm{e}^{T-s} - 1) + 1 = 0.\end{aligned}$$

Thus u satisfies (3.3).

4.2 Linear–quadratic control problem with kernel

Let $\phi : [0,T] \longrightarrow \mathbb{R}$ be continuous. We consider the controlled Volterra-type dynamics

$$X_t = x_0 + \int_0^t \phi(t-s)\big((X_s + \alpha_s)\mathrm{d}s + \mathrm{d}W_s\big),$$

and the control problem

$$V_0 := \sup_{\alpha\in\mathcal{A}} \mathbb{E}\bigg[-\frac{1}{2}\int_0^T (X_s^2 + \alpha_s^2)\mathrm{d}s\bigg].$$

The corresponding lifted problem writes

$$V(t,\mathbf{x}) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}\bigg[-\frac{1}{2}\int_0^T \big((X_s^{t,\mathbf{x},s})^2 + \alpha_s^2\big)\mathrm{d}s\bigg], \tag{4.2}$$

where the flow $\mathbf{X}^{t,\mathbf{x}} := (X^{t,\mathbf{x},s})_{0\leq s\leq T}$ is such that

$$X_r^{t,\mathbf{x},s} = \mathbf{x}^s + \int_t^r \phi(s-t)\big((X_\theta^{t,\mathbf{x},\theta} + \alpha_\theta)\mathrm{d}\theta + \mathrm{d}W_\theta\big),\ \text{for all } s \in [0,T].$$

We easily see that the dynamic programming equation corresponding to this problem is

$$
\begin{aligned}
&-\partial_t u(t,\mathbf{x}) - \frac{1}{2}\iint_{[0,T]^2} D^2_{\mathbf{xx}}u(t,\mathbf{x})(r,s)\phi(r-t)\phi(s-t)\mathrm{d}r\mathrm{d}s + x^t\int_0^T D_{\mathbf{x}}u(t,\mathbf{x})(r)\phi(r-t)\mathrm{d}r \\
&\quad + \frac{1}{2}\left(\int_0^T D_{\mathbf{x}}u(t,\mathbf{x})(r)\phi(r-t)\mathrm{d}r\right)^2 + \frac{(x^t)^2}{2} = 0,
\end{aligned}
\tag{4.3}
$$

with boundary condition $u(T,\cdot) = 0$, as there is no terminal reward. Here we used Lebesgue measure instead of μ as reference measure (since there is no terminal reward) and $\mathbb{L}^2([0,T],\mathrm{d}t)$ as reference space to represent the Fréchet derivatives of u (which does not make a difference in the context of classical solutions).

Our objective is to find a solution u of the form

$$
u(t,\mathbf{x}) = \frac{1}{2}\iint_{[0,T]^2} c(t,r,s)x^r x^s\mathrm{d}r\mathrm{d}s,
$$

where c is a measurable $\mathbb{R}$-valued function defined on $[0,T]^3$, which is symmetric in its last two variables. Let us compute formally the derivatives of u

$$
\partial_t u(t,\mathbf{x}) = \frac{1}{2}\iint_{[0,T]^2}\partial_t c(t,r,s)x^r x^s\mathrm{d}r\mathrm{d}s,\ D_{\mathbf{x}}u(t,\mathbf{x})(r) = \int_0^T c(t,r,s)x^s\mathrm{d}s,\ D^2_{\mathbf{xx}}u(t,\mathbf{x})(r,s) = c(t,r,s).
$$

Observing that $x^t = \int_0^T \delta_t(s)\mathrm{d}s$, where δ_t is the Dirac mass at t, we also compute

$$
x^t\int_0^T D_{\mathbf{x}}u(t,\mathbf{x})(r)\phi(r-t)\mathrm{d}r = \iint_{[0,T]^2} c(t,r,s)\phi(r-t)\delta_t(s)x^r x^s\mathrm{d}r\mathrm{d}s,\ (x^t)^2 = \iint_{[0,T]^2}\delta_t(r)\delta_t(s)x^r x^s\mathrm{d}r\mathrm{d}s,
$$

and

$$
\left(\int_0^T D_{\mathbf{x}}u(t,\mathbf{x})(r)\phi(r-t)\mathrm{d}r\right)^2 = \iint_{[0,T]^2}\left(\int_0^T c(t,r,\theta)\phi(\theta-t)\mathrm{d}\theta\right)\left(\int_0^T c(t,\tau,s)\phi(\tau-t)\mathrm{d}\tau\right)x^r x^s\mathrm{d}r\mathrm{d}s.
$$

Introduce the following notations

$$
(c\star\phi)(t,r) \coloneqq \int_0^T c(t,r,\theta)\phi(\theta-t)\mathrm{d}\theta.
$$

Note that, since c is symmetric in r and s, we also have $(c\star\phi)(t,s) = \int_0^T c(t,\tau,s)\phi(\tau-t)\mathrm{d}\tau$. Plugging all these expressions in the dynamic programming equation (4.3), we see that a, b and c satisfy the following equation

$$
\partial_t c(t,r,s) = -(c\star\phi)(t,r)(c\star\phi)(t,s) - 2\delta_t(s)(c\star\phi)(t,r) - \delta_t(r)\delta_t(s). \tag{4.4}
$$

with terminal conditions $c(T,\cdot,\cdot) = 0$. Intuitively, it makes sense to look for c such that

$$
c(t,r,s) = 0,\ \text{whenever}\ (r,s)\notin[t,T]^2,
$$

since the Volterra dynamics that we consider is 'triangular (*i.e.*, the 'Volterra time is always greater than t for the problem starting at time t). So we expect $c(t,\cdot,\cdot)$ to be zero on $[(t,T]^2)^c$ and to be smooth on $[t,T]^2$. This should roughly be a structure similar to (4.1) from the previous section, where

$$
u(t,\mathbf{x}) = \int_0^T b(t,s)x^s\mu(\mathrm{d}s)
$$

with $b(t,s) = \mathrm{e}^{-s}\mathbf{1}_{[t,T]}(s)$, except that this time the function still depends on t on $[t,T]$, as a consequence of the Volterra structure. Then the size of the Dirac masses in (4.4) accounts for the boundary conditions of c in t, that is the values of $c(t,t,\cdot) = c(t,\cdot,t)$ and $c(t,t,t)$. Finally, our infinite dimensional Riccati system takes the following form

$$
\partial_t c(t,r,s) = -(c\star\phi)(t,r)(c\star\phi)(t,s),\ \text{for all}\ (s,r)\in(t,T]^2, \tag{4.5}
$$

with boundary conditions

$$
\begin{cases}
c(t,t,r) = c(t,r,t) = 2(c\star\phi)(t,r),\ \text{for all}\ r\in(t,T],\\
c(t,t,t) = 1,\\
c(T,\cdot,\cdot) = 0.
\end{cases}
\tag{4.6}
$$

Note that $a = b = 0$ satisfy the two first equalities, and that the third one only depends on c, thus the system becomes

$$\begin{cases} \partial_t c(t,r,s) = -(c \star \phi)(t,r)(c \star \phi)(t,s), \\ c(t,t,r) = c(t,r,t) = 2(c \star \phi)(t,r), \\ c(t,t,t) = 1, \\ c(T,\cdot,\cdot) = 0, \end{cases} \quad \text{for all } (s,r) \in (t,T]^2. \tag{4.7}$$

Remark 4.1. *The above verification also applies to more general linear–quadratic dynamics, similarly to Wang, Yong, and Zhou [40], although we chose to consider simpler coefficients for the sake of clarity. Note also that (4.4) corresponds to* (4.9) *in* [40] *for our choice of coefficients, as well as* Equation (3.2) *from Abi Jaber, Miller, and Pham [2], where the Riccati system is derived for a special kernel through a different technique.*

4.3 Time-inconsistent contract theory

One of our main motivation is related to the works of Hernández and Possamaï [23; 24], which focuses on principal–agent contracting problems in presence of a form of time-inconsistency in the agent's problem. We first recall the setting of the problem and the main results of [23; 24] when the time-inconsistency is due to the presence of a non-exponential discount.

4.3.1 The Agent's and Principal's problems

Given an output process

$$X_t^\alpha := X_0 + \int_0^T \alpha_r dr + W_t,$$

where the effort α takes its value une some compact $[0,\bar{a}] \subset \mathbb{R}$, and a payment ξ given by the Principal, the Agent wants to solve the control problem:

$$V_0^A(\xi) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\big[U_A(0,\xi) - \frac{1}{2}\int_0^T c_r(0,\alpha_r)dr\big],$$

where U_A corresponds to his utility function, c his cost function and $f : \mathbb{R} \to \mathbb{R}$ to the (possibly non-exponential) discount factor. The dynamic version of the Agent's problem, i.e. the control problem seen from any date $t \in [0,T]$, takes the following form:

$$V_t^A(\xi) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\big[U_A(t,\xi) + \frac{1}{2}\int_0^T c_r(t,\alpha_r)dr\big].$$

Clearly, such a problem may not be handled through the traditional dynamic programming approach. Instead, the authors of [23] assume that the Agent plays a leader-follower game with the future versions of himself, therefore looking for a Stackelberg equilibrium, see [23, Definition 2.6]. In particular, they show that

$$V_t^A(\xi) = Y_t^t,$$

where the family of processes $\{Y^s\}_{\{s \in [0,T]\}}$ satisfies the backward system:

$$Y_t^s = U_A(t,\xi) - \int_t^T c_r^*(s, Z_r^r)dr - \int_t^T Z_r^s dW_r,$$

with $c_r^*(z) := c_r(a_r^*(z))$, where a^* corresponds to a Stalckelberg equilibrium. Assuming c is continuous in both r and a and non negative, we observe that, by boundedness of the controls α, c^* takes its values in some compact $[0,\bar{c}]$. In the spirit of [], this system is rewritten in a forward way and the Principal optimizes on ξ by optimizing on Z and Y_0:

$$V^P \quad = \sup\nolimits_{\xi : V^A(\xi) \geq R} \mathbb{E}\big[U_P(X_T,\xi)\big] = \sup\nolimits_{Y_0^\cdot \geq R} V(Y_0^\cdot),$$

where R is the participation constraint V (i.e., the minimal utility guaranteed to the Agent so that he accepts to sign the contract), and V is defined by

$$V(Y_0^\cdot) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\big[U_P(X_T^Z, Y_T^{0,Z})\big], \tag{4.8}$$

where $Y^{s,Z}$ is the forward dynamics:

$$Y_t^{s,Z} = Y_0^s + \int_0^t c_r^*(s, Z_r^r)dr + \int_0^t Z_r^s dW_r, \tag{4.9}$$

and $\mathcal{Z}$ is the set of square integrable doubly indexed processes such that

$$U_A^{(-1)}(s, Y_T^s) = U_A^{(-1)}(0, Y_T^0) \quad \forall\ s \in [0,T], \tag{4.10}$$

where $U_A^{(-1)}$ denotes the inverse of U_A with respect to the second variable. Our goal is to study the control problem (4.8) by using the setting developed in the present paper.

4.3.2 Reformulation as a control problem in H with stochastic target constraints

We assume that the control $\mathbf{Z} := \{Z^s\}_{\{s\in[0,T]\}}$ is such that $\mathbf{Z}_t \in H$ for all $t \in [0,T]$. It is then clear that those conditions ensure that $\mathbf{Y}^{\mathbf{Z}} := \{Y^{Z,s}\}_{\{s\in[0,T]\}}$ takes its values in H as well. We then rewrite the Principal problem:

$$V(\mathbf{y}) = \sup_{\mathbf{Z}\in\mathcal{Z}_H} \mathbb{E}\big[U_P(X, Y_T^{0,\mathbf{Z}})\big], \tag{4.11}$$

where $\mathbf{y} = \mathbf{Y}_0^{\mathbf{Z}}$, $\mathcal{Z}_H$ is the set of square integrable H-valued processes such that (4.10) is satisfied, or equivalently (abusing the notation and denoting a Sobolev function and its continuous representative the same way):

$$Y_T^{0,\mathbf{Z}} = \psi\big(s, Y_T^{s,\mathbf{Z}}\big) \quad \forall\ s \in [0,T],$$

with $\psi(s,y) := U_A\big(0, U_A^{(-1)}(s,y)\big)$ for all $(s,y) \in [0,T]\times\mathbb{R}$. Thus, the Principal must solve a stochastic control problem with stochastic target constraints on a Hilbert space. Note that this constraint is equivalent to

$$\overline{g}(\mathbf{Y}_T^{\mathbf{Z}}) \leq Y_T^{0,\mathbf{Z}} \leq \underline{g}(\mathbf{Y}_T^{\mathbf{Z}}), \tag{4.12}$$

with

$$\underline{g}(\mathbf{x}) = \min_{0\leq s\leq T} \psi(s, \tilde{x}^s) \quad \text{and} \quad \underline{g}(\mathbf{x}) = \max_{0\leq s\leq T} \psi(s, \tilde{x}^s).$$

Reachability set. The first step for the Principal is to determine her reachability set, i.e., the family of sets $\mathcal{V}(t), t\in[0,T]$ in which the state process $\mathbf{Y}^{\mathbf{Z}}$ must lie at each time t so that the target (4.12) can be reached. Clearly, (4.12) involves the two following epigraph-type target problems:

$$\begin{aligned} \underline{w}(t,\mathbf{y}) &:= \inf\{y\in\mathbb{R} : \hat{Y}_T^{0,t,y,\mathbf{Z}} \geq \overline{g}(\mathbf{Y}_T^{t,\mathbf{y},\mathbf{Z}})\}, \\ \overline{w}(t,\mathbf{y}) &:= \sup\{y\in\mathbb{R} : \hat{Y}_T^{0,t,y,\mathbf{Z}} \leq \underline{g}(\mathbf{Y}_T^{t,\mathbf{y},\mathbf{Z})})\}, \end{aligned} \tag{4.13}$$

for $(t,\mathbf{y}) \in [0,T]\times H$, where $\hat{Y}^{0,t,y,\mathbf{Z}}$ is the process following starting from $\hat{Y}_t^{0,t,y,\mathbf{Z}} = y$ following the same SDE as $Y^{0,\mathbf{Z}}$. Indeed, this decoupling is necessary as, in the formulation of epigraph-type target problems in the Appendix, the process $\mathbf{X}$ must not depend on the initial condition of the process Y. However, given the nature of the dynamics of $\mathbf{Y}^{\mathbf{Z}}$, we will see that this does not appear as a serious issue to characterize the Principal's reachability set.
As studied in the Appendix (note that the supremum problem can be straightforwardly re-written as an infimum problem), the value functions are proved to satisfy, under some regularity assumptions, the following equations on $[0,T]\times H$:

$$-\partial_t\underline{w}(t,\mathbf{y}) + \sup_{\mathbf{z}\in\mathcal{N}(t,\mathbf{y},\underline{w}(t,\mathbf{y}),D_{\mathbf{y}}\underline{w}(t,\mathbf{y}))} \Big\{c_t^*(0,z^t) - \langle c_t^*(\cdot,z^t), D_{\mathbf{y}}\underline{w}(t,\mathbf{y})\rangle_H + \frac{1}{2}\langle \mathbf{z}, D^2_{\mathbf{yy}}\underline{w}(t,\mathbf{y})\mathbf{z}\rangle_H\Big\} = 0, \quad \underline{w}(T,\mathbf{y}) = \overline{g}(\mathbf{y}),$$

$$-\partial_t\overline{w}(t,\mathbf{y}) + \inf_{\mathbf{z}\in\mathcal{N}(t,\mathbf{y},\overline{w}(t,\mathbf{y}),D_{\mathbf{x}}\overline{w}(t,\mathbf{y}))} \Big\{c_t^*(0,z^t) - \langle c_t^*(\cdot,z^t), D_{\mathbf{y}}\overline{w}(t,\mathbf{y})\rangle_H + \frac{1}{2}\langle \mathbf{z}, D^2_{\mathbf{yy}}\overline{w}(t,\mathbf{y})\mathbf{z}\rangle_H\Big\} = 0, \quad \overline{w}(T,\mathbf{y}) = \underline{g}(\mathbf{y}),$$

with

$$\mathcal{N}(t,\mathbf{y},\mathbf{p}) := \{\mathbf{Z}\in H : Z^0 - \langle \mathbf{Z},\mathbf{p}\rangle_H = 0\} = \{\mathbf{Z}\in H : \langle \mathbf{Z}, \mathbf{v}_0 - \mathbf{p}\rangle_H = 0\},$$

where $\mathbf{v}_0$ is the element of H such that $\langle \mathbf{v}_0, \mathbf{x}\rangle_H = \tilde{x}^0$ for all $\mathbf{x}\in H$.
We now informally describe how we expect the reachability set to look like. First, we clearly have the following inclusion:

$$\mathcal{V}(t) \subset \{\mathbf{y}\in H : \underline{w}(t,\mathbf{y}) \leq y^0 \leq \overline{w}(t,\mathbf{y})\}.$$

Indeed, if $\mathbf{y}\in\mathcal{V}(t)$, then there exists $\mathbf{Z}$ such that $\overline{g}(\mathbf{Y}_T^{t,\mathbf{y},\mathbf{Z}}) \leq Y_T^{0,t,\mathbf{y},\mathbf{Z}} \leq \underline{g}(\mathbf{Y}_T^{t,\mathbf{y},\mathbf{Z}})$. Note that, as the $\{Y^{s,t,\mathbf{y},\mathbf{Z}}\}_{\{0\leq s\leq T\}}$ only interact through the control $\mathbf{Z}$, we have in fact $Y^{0,t,\mathbf{y},\mathbf{Z}} = \hat{Y}^{0,t,y^0,\mathbf{Z}}$, $\mathbb{P}$-a.s. Therefore, $\mathbf{Z}$ is admissible for both stochastic target problems (4.13), and therefore $y^0 \in [\underline{w}(t,\mathbf{y}), \overline{w}(t,\mathbf{y})]$. The converse inclusion is more challenging, and we refer to Hernández, Santibáñez, Hubert, and Possamaï [25, Lemma 5.3] for more detail about how it can be proved in a finite dimensional context. Assuming the same can be done, we shall consider in what follows that the closure of the Principal's reachability set is equal to the previous set:

$$\mathrm{cl}\big(\mathcal{V}(t)\big) = \{\mathbf{y}\in H : \underline{w}(t,\mathbf{y}) \leq y^0 \leq \overline{w}(t,\mathbf{y})\}. \tag{4.14}$$

Dynamic programming equation. Given this characterization of the reachability set, the Principal's problem may be reformulated as a state-constrained control problem:

$$V(t,x,\mathbf{y}) := \sup_{\mathbf{Z}: \mathbf{Y}_r^{t,\mathbf{y},\mathbf{Z}} \in \mathcal{V}(r)\ \forall r\in[t,T]} \mathbb{E}\big[U_P(X_T^{t,x,\mathbf{Z}}, Y_T^{0,\mathbf{Z}})\big],$$

with $(X^{t,x,\mathbf{Z}}, \mathbf{Y}^{t,\mathbf{Y},\mathbf{Z}})$ following the dynamics:

$$\begin{aligned} X_s^{t,x,\mathbf{Z}} &= x + \int_t^s a_r^*(Z_r^r)dr + W_s - W_t \\ \mathbf{Y}^{t,\mathbf{Y},\mathbf{Z}} &= \mathbf{y} + \int_t^s c_r^*(\cdot, Z_r^r)dr + \int_t^s \mathbf{Z}_r dW_r. \end{aligned}$$

On the model of [25], who relies on the previous work of Bouchard, Élie, and Imbert [8], we then formally derive the Principal's dynamic programming equation:

$$\begin{cases} -\partial_t V(t,x,\mathbf{y}) + F(t,x,\mathbf{y},\partial_x V(t,x,\mathbf{y}), \partial^2_{xx}V(t,x,\mathbf{y}), D_{\mathbf{y}}V(t,x,\mathbf{y}), D^2_{\mathbf{yy}}V(t,x,\mathbf{y}), \partial_x D_{\mathbf{y}}V(t,x,\mathbf{y})) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } \mathbf{y} \text{ s.t. } \underline{w}(t,\mathbf{y}) < y^0 < \overline{w}(t,\mathbf{y}), \\ -\partial_t V(t,x,\mathbf{y}) + \underline{F}(t,x,\mathbf{y},\partial_x V(t,x,\mathbf{y}), \partial^2_{xx}V(t,x,\mathbf{y}), D_{\mathbf{y}}V(t,x,\mathbf{y}), D^2_{\mathbf{yy}}V(t,x,\mathbf{y}), \partial_x D_{\mathbf{y}}V(t,x,\mathbf{y})) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } \mathbf{y} \text{ s.t. } \underline{w}(t,\mathbf{y}) = y^0, \\ -\partial_t V(t,x,\mathbf{y}) + \overline{F}(t,x,\mathbf{y},\partial_x V(t,x,\mathbf{y}), \partial^2_{xx}V(t,x,\mathbf{y}), D_{\mathbf{y}}V(t,x,\mathbf{y}), D^2_{\mathbf{yy}}V(t,x,\mathbf{y}), \partial_x D_{\mathbf{y}}V(t,x,\mathbf{y})) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for } \mathbf{y} \text{ s.t. } y^0 = \overline{w}(t,\mathbf{y}), \\ V(T,\cdot) = U_P, \end{cases} \tag{4.15}$$

where

$$\begin{aligned} F(t,x,\mathbf{y},p,A,\mathbf{p},\mathbf{A},\mathbf{q}) &= \sup_{\mathbf{z}\in H} \Big\{ a_t^*(z^t)p + \langle c_t^*(\cdot,z^t), \mathbf{p}\rangle_H + \frac{1}{2}\big(A + \langle \mathbf{z}, \mathbf{A}\mathbf{z}\rangle_H + 2\langle \mathbf{z},\mathbf{q}\rangle_H\big)\Big\}, \\ \underline{F}(t,x,\mathbf{y},p,A,\mathbf{p},\mathbf{A},\mathbf{q}) &= \sup_{\mathbf{z}\in \underline{H}(t,\mathbf{y},\underline{w})} \Big\{ a_t^*(z^t)p + \langle c_t^*(\cdot,z^t), \mathbf{p}\rangle_H + \frac{1}{2}\big(A + \langle \mathbf{z}, \mathbf{A}\mathbf{z}\rangle_H + 2\langle \mathbf{z},\mathbf{q}\rangle_H\big)\Big\}, \\ \overline{F}(t,x,\mathbf{y},p,A,\mathbf{p},\mathbf{A},\mathbf{q}) &= \sup_{\mathbf{z}\in \overline{H}(t,\mathbf{y},\overline{w})} \Big\{ a_t^*(z^t)p + \langle c_t^*(\cdot,z^t), \mathbf{p}\rangle_H + \frac{1}{2}\big(A + \langle \mathbf{z}, \mathbf{A}\mathbf{z}\rangle_H + 2\langle \mathbf{z},\mathbf{q}\rangle_H\big)\Big\}, \end{aligned}$$

with the sets $\underline{H}$ and $\overline{H}$ defined by:

$$\begin{aligned} \underline{H}(t,\mathbf{y},w) &:= \{\mathbf{z}\in H : \langle \mathbf{z}, D_{\mathbf{y}}w(t,\mathbf{y})\rangle_H - y^0 = 0 \text{ and } c_t^*(0,z^t) - \langle c_t^*(\cdot,z^t), D_{\mathbf{y}}w(t,\mathbf{y})\rangle_H + \frac{1}{2}\langle \mathbf{z}, D^2_{\mathbf{yy}}w(t,\mathbf{y})\mathbf{z}\rangle_H \geq 0\}, \\ \overline{H}(t,\mathbf{y},w) &:= \{\mathbf{z}\in H : \langle \mathbf{z}, D_{\mathbf{y}}w(t,\mathbf{y})\rangle_H - y^0 = 0 \text{ and } c_t^*(0,z^t) - \langle c_t^*(\cdot,z^t), D_{\mathbf{y}}w(t,\mathbf{y})\rangle_H + \frac{1}{2}\langle \mathbf{z}, D^2_{\mathbf{yy}}w(t,\mathbf{y})\mathbf{z}\rangle_H \leq 0\} \end{aligned}$$

for all $(t,\mathbf{y}) \in [0,T]\times H$ and smooth $w : [0,T]\times H \to \mathbb{R}$.

Remark 4.2. (i) *We insist on the fact that the derivation of the above PDE is very informal. The characterization of V as the (unique) viscosity solution to* (4.15) *is left for further research.*
(ii) *Assuming smoothness of V, $\underline{w}$ and $\overline{w}$, existence of optimal controls and that the reachability set of the Principal indeed writes as* (4.14) *and following the methodology of [25] (the dimension does not play an important role as the proof mostly relies on an application of Itô's formula), we should be able to provide a verification theorem for the Principal's problem, i.e., that if v is a classical solution of* (4.15)*, then it matches the Principal's value function.*

4.3.3 Study for a special discount factor

We end the discussion on the Principal-Agent problem with an example where a reduction of dimension can be obtained. Consider the case where the Agent's time-inconsistency only comes from the presence of a non-exponential discount factor:

$$U_A(s,\xi) = f(T-s)U_A(\xi),\ c_t(s,a) = F(t-s)c_t(a).$$

Moreover, we assume that the function f takes the form:

$$f(t) = \sum_{k=1}^N \beta_k e^{-\rho_k t},$$

for some $N\in\mathbb{N}^*$, where the sequences β and ρ are non-negative such that $\sum_{k=1}^N \beta_k = 1$ and $\rho_k \neq \rho_j$ for all $k\neq j$. We then have the following inclusion:

Lemma 4.3. *For all $k \in \{1, \ldots, N\}$, denote: $\phi_k(s) = \beta_k e^{\rho_k s}$. Then we have, for all $t \in [0, T]$:*

$$\mathbf{Y}_t^{\mathbf{Z}}, \mathbf{Z}_t \in \mathrm{Vect}\big\{\phi_1, \ldots, \phi_{\mathrm{N}}\big\}.$$

Proof. First observe that the stochastic target constraint (4.10) writes in this context:

$$Y_T^{s,\mathbf{Z}} = \frac{f(T-s)}{f(T)} Y_T^{0,\mathbf{Z}} = \frac{1}{f(T)} \sum_{k=1}^N \phi_k(s) e^{-\rho_k T} Y_T^{0,\mathbf{Z}},$$

for all $s \in [0, T]$. Taking the conditional expectation with respect to $\mathcal{F}_t$ in the above equality, we obtain:

$$Y_t^{s,\mathbf{Z}} + \mathbb{E}\Big[\int_t^T f(T-s) c_r^*(Z_r^r) dr | \mathcal{F}_t\Big] = \frac{1}{f(T)} \sum_{k=1}^N \phi_k(s) e^{-\rho_k T} \Big(Y_t^{0,\mathbf{Z}} + \mathbb{E}\Big[\int_t^T f(T) c_r^*(Z_r^r) dr | \mathcal{F}_t\Big]\Big),$$

which means that $\mathbf{Y}_t^{\mathbf{Z}} \in \mathrm{Vect}\big\{\phi_1, \ldots, \phi_{\mathrm{N}}\big\}$ since $f(T-\cdot) \in \mathrm{Vect}\big\{\phi_1, \ldots, \phi_{\mathrm{N}}\big\}$. Furthermore, as both $\mathbf{Y}^{\mathbf{Z}}$ and its drift term lie in this space, we have

$$\int_0^t \mathbf{Z}_r dW_r \in \mathrm{Vect}\big\{\phi_1, \ldots, \phi_{\mathrm{N}}\big\} \text{ for all } t \in [0, T],$$

which in turns implies that $\mathbf{Y}_t^{\mathbf{Z}} \in \mathrm{Vect}\big\{\phi_1, \ldots, \phi_{\mathrm{N}}\big\}$ for all t. ■

The main consequence of Lemma 4.3 is that the Principal's problem becomes infinite dimensional. Indeed, admissible controls write:

$$Z_\cdot^s = \sum_{k=1}^N \phi_k(s) \tilde{Z}_\cdot^k,$$

where the $\tilde{Z}_\cdot^k$, $k = 1, \ldots, N$, are $\mathbb{R}$-valued adapted processes. Similarly, we have $\mathbf{Y} = \sum_{k=1}^N \phi_k \tilde{Y}^k$, where:

$$\tilde{Y}_t^k = \tilde{Y}_0^k + \int_0^t e^{-\rho_k r} c_r^*\Big(\sum_{k=1}^N \phi_k(r) \tilde{Z}_r\Big) dr + \int_0^t \tilde{Z}_r dW_t$$

for all $k \in \{1, \ldots, N\}$, observing that $Z_r^r = \sum_{k=1}^N \phi_k(r) \tilde{Z}_r$. Then, decomposing $f(T-\cdot)$ in the basis $(\phi_1, \ldots, \phi_N)$, we see that the stochastic target constraint (4.10) reformulates as follows:

$$\tilde{Y}_T^k = \beta_k \frac{e^{-\rho_k T}}{f(T)} \sum_{k=1}^N \beta_k \tilde{Y}_T^k \quad \text{for all } k \in \{1, \ldots, N\}, \tag{4.16}$$

as $\mathbf{Y}_T^0 = \sum_{k=1}^N \beta_k \phi_k(0) \tilde{Y}_T^k = \sum_{k=1}^N \beta_k \tilde{Y}_T^k$. We easily see that this constraint means that the vector $\tilde{Y}_T := (\tilde{Y}_T^1, \ldots, Y_T^N)$ must belong to the line $\mathcal{D}$ in $\mathbb{R}^N$ defined by the system of equations:

$$y_1 = \frac{\beta_1}{\beta_k} e^{(\rho_k - \rho_1)T} y_k \quad \text{for all } k \in \{2, \ldots, N\},$$

or, equivalently, that

$$\max_{k \in \{2, \ldots, N\}} \frac{\beta_1}{\beta_k} e^{(\rho_k - \rho_1)T} y_k \leq y_1 \leq \min_{k \in \{2, \ldots, N\}} \frac{\beta_1}{\beta_k} e^{(\rho_k - \rho_1)T} y_k,$$

which writes again as the combination of two epigraph-type constraints. Then, the Principal must solve the finite dimensional control problem under stochastic target constraint:

$$\sup_{\tilde{Z}: \tilde{Y}_T \in \mathcal{D}} \mathbb{E}\Big[U_P\Big(X, \sum_{k=1}^N \beta_k \tilde{Y}_T^k\Big)\Big],$$

Note that this falls under the setting of Hernández et al. [25], who have been able to derive the dynamic programming equation characterizing this problem.

5 Markovian representation and approximation

In this section, we discuss how our framework provides a natural Markovian approximation for the Volterra-type dynamics (1.1). For the sake of clarity, we omit the dependence on the control α for now. When b and σ are as in (1.1), we recall that the Volterra-type process X is related to the infinite dimensional process $\mathbf{X}$ from (2.3) in the following way

$$X_t = \phi(t, \mathbf{X}_t), \text{ for all } t \in [0,T],$$

where the mapping $\phi : [0,T] \times H \longrightarrow \mathbb{R}$ is defined by

$$\phi(t, \mathbf{x}) \coloneqq \tilde{x}^t, \text{ for all } (t, \mathbf{x}) \in [0,T] \times H.$$

By (2.2), the mapping $\phi(t, \cdot)$ is a continuous linear form on H for all $t \in [0,T]$. Therefore, by Riesz's representation theorem, there exists an H-valued mapping $t \longmapsto v_t$ such that

$$\phi(t, \mathbf{x}) = \langle v_t, \mathbf{x} \rangle_H \text{ for all } (t, \mathbf{x}) \in [0,T] \times H.$$

Note that v_t is a Sobolev solution of the equation

$$v_t - \partial_s^2 v_t = \delta_t, \text{ for all } t \in [0,T]. \tag{5.1}$$

Note that t is only a parameter here. Solving the equation on $(0,t)$ and (t,T), and using the continuity condition of $s \longmapsto v_t^s$ in t, we see that v has the form

$$v_t^s = \frac{1}{2}\big(A\mathrm{e}^s + B\mathrm{e}^{-s}\big) + \frac{1}{2}\big(\mathrm{e}^{s-t} - \mathrm{e}^{-(s-t)}\big)\mathbf{1}_{\{s \geq t\}}, \ (A,B) \in \mathbb{R}^2. \tag{5.2}$$

Let $(e_k)_{k \in \mathbb{N}^\star}$ be an orthonormal basis of H. Denote for any $t \in [0,T]$ and any $k \in \mathbb{N}^\star$ by v_t^k and X_t^k the projections of v_t and $\mathbf{X}_t$ on e_k. We have

$$X_t = \sum_{k=1}^{\infty} v_t^k X_t^k. \tag{5.3}$$

For all $k \in \mathbb{N}^\star$, X^k is a diffusion process solving the SDE

$$X_t^k = x_k + \int_0^T b_r^k(X_r)\mathrm{d}r + \int_0^T \sigma_r^k(X_r)\mathrm{d}W_r,$$

where

$$x_k \coloneqq \langle \mathbf{x}, e_k \rangle_H, \ b_t^k(x) \coloneqq \langle b_t(x), e_k \rangle_H, \text{ and } \sigma_t^k(x) \coloneqq \langle \sigma_t(x), e_k \rangle_H, \ \forall (x,k) \in \mathbb{R} \times \mathbb{N}^\star.$$

Note also that for any $k \in \mathbb{N}^\star$, $t \longmapsto v_t^k$ is C^1. Indeed, we have

$$\begin{aligned} v_t^k = \langle v_t, e_k \rangle_H = \int_0^T v_t^s e_k^s \mathrm{d}s + \int_0^T \partial_s v_t^s \partial_s e_k^s \mathrm{d}s = \frac{1}{2}\int_0^T \big((A\mathrm{e}^s + B\mathrm{e}^{-s})e_k^s + (A\mathrm{e}^s - B\mathrm{e}^{-s})\partial_s e_k^s\big)\mathrm{d}s \\ + \frac{1}{2}\int_t^T \big((\mathrm{e}^{s-t} + \mathrm{e}^{-(s-t)})e_k^s + (\mathrm{e}^{s-t} - \mathrm{e}^{-(s-t)})\partial_s e_k^s\big)\mathrm{d}s, \end{aligned}$$

which is clearly C^1 in t. Thus, we may formally differentiate (5.3) to obtain

$$\mathrm{d}X_t = \sum_{k=1}^{\infty} \partial_t v_t^k X_t^k \mathrm{d}t + v_t^k \mathrm{d}X_t^k.$$

Then, denoting $X^0 \coloneqq X$, we see that the infinite dimensional system $(X^k)_{k \in \mathbb{N}^\star}$ follows the Markovian dynamics

$$\begin{cases} X_t^0 = x_0 + \displaystyle\int_0^t \Big(\sum_{k=1}^{\infty} \partial_t v_r^k X_r^k + v_r^k b_r^k(X_r^0)\Big)\mathrm{d}r + \int_0^t \sum_{k=1}^{\infty} v_r^k \sigma_r^k(X_r^0)\mathrm{d}W_r, \\ X_t^k = x_k + \displaystyle\int_0^T b_r^k(X_r^0)\mathrm{d}r + \int_0^T \sigma_r^k(X_r^0)\mathrm{d}W_r, \ k \in \mathbb{N}^\star. \end{cases} \tag{5.4}$$

Of course, a natural finite-dimensional approximation is obtained by simply truncating the sums in the above dynamics; thus, for $n \in \mathbb{N}^\star$, we may consider the $(n+1)$-dimensional dynamics

$$\begin{cases} X_t^{0,n} = x_0 + \displaystyle\int_0^t \bigg(\sum_{k=1}^n \partial_t v_r^k X_r^{k,n} + v_r^k b_r^k(X_r^{0,n}) \bigg) \mathrm{d}r + \int_0^t \sum_{k=1}^n v_r^k \sigma_r^k(X_r^{0,n}) \mathrm{d}W_r, \\ X_t^{k,n} = x_k + \displaystyle\int_0^T b_r^k(X_r^{0,n}) \mathrm{d}r + \int_0^T \sigma_r^k(X_r^{0,n}) \mathrm{d}W_r,\ k \in \mathbb{N}^\star, \end{cases} \tag{5.5}$$

which is a standard finite dimensional SDE.

Proposition 5.1. *We have*

$$\lim_{n\to\infty} \mathbb{E}\bigg[\sup_{t\in[0,T]} \big| X_t^{0,n} - \bar{X}_t^n \big|^2 \bigg] = 0, \text{ and } \lim_{n\to\infty} \mathbb{E}\big[|X_t^{0,n} - X_t^0|^2 \big] = 0,\ \forall t \in [0,T].$$

Proof. Introduce for any $n \in \mathbb{N}^\star$

$$\bar{X}_t^n \coloneqq \sum_{k=1}^n v_t^k X_t^k, \text{ and } R_t^n \coloneqq \sum_{k=n+1}^\infty v_t^k X_t^k.$$

Recalling (5.3), we have for any $t \in [0,T]$ and any $n \in \mathbb{N}^\star$

$$|X_t^{0,n} - X_t^0|^2 \leq 2|X_t^{0,n} - \bar{X}_t^n|^2 + 2|R_t^n|^2. \tag{5.6}$$

Since $X_t^{0,n} = \sum_{k=1}^n v_t^k X_t^{k,n}$, we have for any $t \in [0,T]$ and any $n \in \mathbb{N}^\star$

$$X_t^{0,n} - \bar{X}_t^n = \int_0^t \bigg(\sum_{k=1}^n v_t^k \big(b_s^k(X_s^{0,n}) - b_s^k(X_s^0) \big) \bigg) \mathrm{d}s + \int_0^t \bigg(\sum_{k=1}^n v_t^k \big(\sigma_s^k(X_s^{0,n}) - \sigma_s^k(X_s^0) \big) \bigg) \mathrm{d}W_s.$$

By using Cauchy–Schwarz's inequality and Doob's inequalities, we have for any $t \in [0,T]$ and any $n \in \mathbb{N}^\star$

$$\begin{aligned} \mathbb{E}\bigg[\sup_{u\in[0,t]} \big| X_u^{0,n} - \bar{X}_u^n \big|^2 \bigg] &\leq 2T \int_0^t \mathbb{E}\bigg[\bigg(\sum_{k=1}^n v_t^k \big(b_s^k(X_s^{0,n}) - b_s^k(X_s^0) \big) \bigg)^2 \bigg] \mathrm{d}s + 8 \int_0^t \mathbb{E}\bigg[\bigg(\sum_{k=1}^n v_t^k \big(\sigma_s^k(X_s^{0,n}) - \sigma_s^k(X_s^0) \big) \bigg)^2 \bigg] \mathrm{d}s \\ &\leq 2T \int_0^t \bigg(\sum_{k=1}^n |v_t^k|^2 \bigg) \mathbb{E}\bigg[\bigg(\sum_{k=1}^n |b_s^k(X_s^{0,n}) - b_s^k(X_s^0)|^2 \bigg) \bigg] \mathrm{d}s \\ &\quad + 8 \int_0^t \bigg(\sum_{k=1}^n |v_t^k|^2 \bigg) \mathbb{E}\bigg[\bigg(\sum_{k=1}^n |\sigma_s^k(X_s^{0,n}) - \sigma_s^k(X_s^0)|^2 \bigg) \bigg] \mathrm{d}s \\ &\leq 2T \int_0^t \|v_t\|_H^2 \mathbb{E}\big[\|b_s(\cdot, X_s^{0,n}) - b_s(\cdot, X_s^0)\|_H^2 \big] \mathrm{d}s + 8 \int_0^t \|v_t\|_H^2 \mathbb{E}\big[\|\sigma_s(\cdot, X_s^{0,n}) - \sigma_s(\cdot, X_s^0)\|_H^2 \big] \mathrm{d}s. \end{aligned}$$

Recalling that b and σ are Lipschitz-continuous as H-valued functions and the estimate from Equation (5.6), we have

$$\mathbb{E}\bigg[\sup_{u\in[0,t]} \big| X_u^{0,n} - \bar{X}_u^n \big|^2 \bigg] \leq 4(T+4)L^2 \|v^\star\|_H^2 \int_0^t \Big(\mathbb{E}\big[|X_s^{0,n} - \bar{X}_s^n|^2 \big] + \mathbb{E}\big[|R_s^n|^2 \big] \Big) \mathrm{d}s, \tag{5.7}$$

with $\|v^\star\|_H \coloneqq \sup_{t\in[0,T]} \|v_t\|_H$. Therefore, by Grönwall's lemma, we have

$$\mathbb{E}\bigg[\sup_{u\in[0,T]} \big| X_u^{0,n} - \bar{X}_u^n \big|^2 \bigg] \leq (\int_0^T \mathbb{E}\big[|R_s^n|^2 \big] \mathrm{d}s \exp\big(4(T+4)L^2 T \|v^\star\|_H^2 \big). \tag{5.8}$$

Notice that

$$\big| R_s^n \big|^2 \leq \bigg(\sum_{k=n+1}^\infty |v_t^k|^2 \bigg) \bigg(\sum_{k=n+1}^\infty |X_t^k|^2 \bigg) \leq \|v_t\|_H^2 \|\mathbf{X}_t\|_H^2,$$

and that $R_s^n \underset{n\to\infty}{\longrightarrow} 0$, $\mathrm{d}t \otimes \mathbb{P}$–a.e., so that we may apply the dominated convergence theorem to conclude that

$$\int_0^T \mathbb{E}\big[|R_s^n|^2 \big] \mathrm{d}s \underset{n\to\infty}{\longrightarrow} 0.$$

We then combine this limit to obtain the first result, and plug it into (5.6) to obtain the second one.

■

Remark 5.2. *The convergence rate of our Markovian approximation typically writes as a function of* $R_t^n = \sum_{k=n+1}^{\infty} v_t^k X_t^k$, *which strongly depends on the chosen Hilbertian base* $(e_k)_{k\in\mathbb{N}^\star}$. *Therefore, finding the optimal convergence rate of our approximation amounts to finding the best base on which projecting the first* n *coordinates of the state process* $\mathbf{X}$. *This question is far beyond the scope of the present paper, and we shall leave it for further research. Note that several previous papers have already been concerned with multidimensional Markovian approximations of Volterra-type dynamics (in the case of monotone kernels), see* Abi Jaber and El Euch [1], Abi Jaber, Miller, and Pham [2], Harms [22], Alfonsi and Kebaier [5] *or* Bayer and Breneis [7].

6 Comparison with other results in the literature

6.1 Lifting approach

In this section, we discuss how our methodology completes pre-existing results regarding the control of stochastic Volterra integral equations. It is common in the literature (see e.g. Abi Jaber, Miller, and Pham [2] or Hamaguchi [20]) to assume that the control process has the following dynamics

$$X_t^\alpha = x + \int_0^t K(t-r)\big(b(X_r^\alpha)\mathrm{d}r + \sigma(X_r^\alpha)\mathrm{d}W_r\big), \tag{6.1}$$

where the kernel K writes $K(t) \coloneqq \int_{\mathbb{R}} \mathrm{e}^{-\theta t}\mu(d\theta)$ for some signed measure μ. Note that such kernels may be singular at 0. In [12], di Nunno and Giordano generalise this structure by writing

$$K(t) = \langle g, \mathcal{S}_t \nu\rangle_{Y\times Y^\star}, \tag{6.2}$$

where Y is a UMD Banach space, $Y^\star$ its dual, $\mathcal{S}$ a semi-group acting on $Y^\star$ and g, ν two elements in Y and $Y^\star$ respectively. We easily see that the examples of [2] and [20] are covered by this structure. Then, switching the integrals in $\mathrm{d}r$ and $\mathrm{d}W_r$ and the duality bracket $\langle\cdot,\cdot\rangle_{Y\times Y^\star}$, we can write

$$X_t^\alpha = x + \langle g, \mathbf{X}_t^\alpha\rangle_{Y\times Y^\star},$$

where $\mathbf{X}^\alpha$ is a $Y^\star$-valued process satisfying the infinite dimensional Markovian SDE

$$\mathbf{X}_t^\alpha = \int_0^t \mathcal{A}^\star \mathbf{X}_r \mathrm{d}r + \int_0^t \nu b\big(\langle g, \mathbf{X}_r^\alpha\rangle_{Y\times Y^\star}\big)\mathrm{d}r + \int_0^t \nu\sigma\big(\langle g, \mathbf{X}_r^\alpha\rangle_{Y\times Y^\star}\big)\mathrm{d}W_r,$$

thanks to the semi-group structure of $\mathcal{S}$. The problem of controlling X^α is then reduced to the problem of controlling the infinite dimensional Markovian dynamics $\mathbf{X}^\alpha$.

Our approach generalises this reduction for the case of regular kernels. In particular, we do not require any semi-group structure. Recall our general Volterra-type dynamics

$$X_t^\alpha = x + \int_0^t b_r(t, X_r^\alpha)\mathrm{d}r + \sigma_r(t, X_r^\alpha)\mathrm{d}W_r.$$

Then, assuming that b and σ have Sobolev regularity in the 'Volterra time' t, we may write

$$b_r(t,x) = \langle b_r, (\cdot, x), v_t\rangle_H, \text{ and } \sigma_r(t,x) = \langle \sigma_r, (\cdot, x), v_t\rangle_H,$$

where v_t is defined by (5.2) for all $t \in [0,T]$. Then, observing that $x = \langle \mathbf{p}(x), v_t\rangle_H$, we obtain

$$X_t^\alpha = \langle \mathbf{X}_t^\alpha, v_t\rangle_H,$$

where $\mathbf{X}^\alpha$ follows the H-valued Markovian SDE:

$$\mathbf{X}_t^\alpha = \mathbf{p}(x) + \int_0^t b_r\big(\cdot, \langle \mathbf{X}_r^\alpha, v_r\rangle_H\big)\mathrm{d}r + \int_0^t \sigma_r\big(\cdot, \langle \mathbf{X}_r^\alpha, v_r\rangle_H\big)\mathrm{d}W_r.$$

6.2 PDE approach

We now mention other works connecting Volterra dynamics to partial differential equations, which are often used jointly to a lifting approach. We first mention the contribution of Viens and Zhang [38], whose methodology is in spirit the closest to ours. Their idea is the following: given a process X of the form

$$X_t = \int_0^t K(r,t)\mathrm{d}W_r, \tag{6.3}$$

with K a possibly singular kernel, one wants to find a PDE characterising the process

$$Y_t := \mathbb{E}\bigg[\int_t^T f(r,X_r)\mathrm{d}r + g(X_T)\bigg|\mathcal{F}_t\bigg]. \tag{6.4}$$

As usual, the trick is to find a function $u : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ and an adapted process $\tilde{X}$ so that $Y_t = u\big(t,\tilde{X}_t\big)$. Of course, the connection between the stochastic representation above and the PDE is derived through an appropriate form of Itô's formula, for which a semi-martingale structure on $\tilde{X}$ is necessary. Since X defined in (6.3) is obviously not a semi-martingale, the authors of [38] introduce the family of auxiliary processes Θ defined by

$$\Theta_s^t := \int_0^t K(r,s)\mathrm{d}W_r, \text{ for all } s \geq t, \tag{6.5}$$

and prove that

$$Y_t = u(t, X\otimes_t \Theta^t),$$

where $(X\otimes_t \Theta^t)_s := X_s\mathbf{1}_{\{s<t\}} + \Theta_s^t\mathbf{1}_{\{s\geq t\}}$ and u satisfies the path-dependent PDE

$$\partial_t u(t,\omega) + \frac{1}{2}\partial^2_{\omega\omega}u(t,\omega)(K(t,\cdot),K(t,\cdot)) + f(t,\omega) = 0,\ u(t,\omega) = g(T,\omega), \tag{6.6}$$

for all $(t,\omega)\in[0,T]\times C^0([0,T],\mathbb{R})$, where the path derivative is defined in the spirit of Dupire [13], on the space of càdlàg paths. This characterisation assumes that the derivatives of the function u are well defined. This approach has been extended by Wang, Yong, and Zhang in [39] to the case of Volterra-type forward–backward SDEs. In [40], Wang, Yong, and Zhou uses this dynamic programming equation to characterise the value function of a linear–quadratic problem by a system of path-dependent Riccati equations.

Let us now compare this approach with our work. We easily see that the family of random variables $(\Theta_s^t)_{s\in[t,T]}$ corresponds to our $(X_t^s)_{s\in[t,T]}$, the only difference being that the domain of (t,s) in [38] is triangular (*i.e.*, one requires $s>t$) due to the potential singularity of K, whereas it is rectangular in our framework. Furthermore, for fixed t, we easily see that $u(t,\omega) = u(t,\omega\mathbf{1}_{[t,T]})$ in this example. Therefore, introducing Θ defined by (6.5) is almost equivalent to our lifting from X to $\mathbf{X} = (X^s)_{s\in[0,T]}$, with the difference that the space where the new state process takes its values is not the same. In [38], this would be the Banach space of càdlàg paths, whereas we chose the Hilbert space of Sobolev functions on $[0,T]$. Although this forces us to have stronger regularity assumptions on the coefficients of our dynamics, this dramatically reduces the need for regularity on u, as we can resort to the standard theory of viscosity solution on Hilbert space to derive our dynamics programming equation.

We mention again the recent contribution of di Nunno and Giordano [12], who characterise the solution of a Volterra control problem by means of a backward SDE and as the mild solution to the corresponding dynamic programming equation. More precisely, as highlighted in Section 3.3, they consider the controlled dynamics (3.7), with the extra assumption that the kernel K has the structure (6.2). They are then able to prove that, given that the value function has a first order Gâteaux derivative, the problem is characterised by mean of a semi-linear PDE on an UMD Banach space, satisfied in the sense of mild solutions. In our context, since we operate our lifting in a Hilbert space, this connection between backward SDEs and mild solution is standard (see *e.g.* Briand and Confortola [10]), and thus our stochastic representation of the value function in Proposition 3.5 immediately implies that (3.3) is satisfied in the mild sense.

7 Case of singular kernels

In this section, we discuss how we could adapt our approach to the case of singular kernels. Consider for example that the (non-lifted) state process writes

$$X_t = \int_0^t K(t-r)\mathrm{d}W_r,\ t\in[0,T],$$

with $K:(0,T]\longrightarrow\mathbb{R}$ such that $K(t)\longrightarrow_{t\to 0}\infty$ and $K\in\mathbb{L}^2((0,T],\mathrm{d}t)$. Control problems involving this type of dynamics naturally arise in rough volatility models, see *e.g.* the portfolio optimisation problem studied by Fouque and Hu [17]. In our context, our lifting writes $\mathbf{X}_t := \{X_t^s\}_{(t,s)\in\Delta}$, with

$$X_t^s = \int_0^t K(s-r)dW_r,\ t\in[0,T],\ \text{and}\ \Delta := \{(t,s)\in[0,T]^2 : t<s\},$$

as the kernel K is not defined on non-positive times. Since our lifting requires the $\{X_t^s\}_{\{t<s\leq T\}}$ to all belong to the same space, we could consider to artificially extend K on non-positive times, so that (t,s) belong to the rectangular space $[0,T]^2$ instead of the triangular space Δ. Then we have

$$X_t^s = \left(\int_0^s K(s-r)\mathrm{d}W_r\right)\mathbf{1}_{\{0\leq s\leq t\}} + \left(\int_0^t K(s-r)\mathrm{d}W_r\right)\mathbf{1}_{\{s>t\}} = X_s\mathbf{1}_{\{0\leq s\leq t\}} + \left(\int_0^t K(s-r)\mathrm{d}W_r\right)\mathbf{1}_{\{s>t\}},$$

which corresponds to the approach proposed by Viens and Zhang in [38]. However, although $s\longmapsto X_t^s$ is continuous a.s., the possible roughness of $X.$ also prevents us to embed this lifting in a Sobolev space. The lifted control problem should then be studied in the Banach space of continuous paths on $[0,T]$. One could use the functional Itô's formula of [38] (based on Gâteaux derivatives) to derive the corresponding dynamic programming equation for smooth value functions; however, finding the appropriate notion of viscosity solutions for this problem would remain challenging, as the absence of semi-martingale structure prevents use from using the theory developed by Ekren, Keller, Touzi, and Zhang [14], ans is therefore left for future research.

A A class of stochastic target problems in Hilbert spaces

We propose an extension of the results of Soner and Touzi [37] to an infinite dimensional setting. In this paragraph, H denotes an arbitrary Hilbert space, and W a standard $\mathbb{R}$-valued Brownian motion. Let $\mathbf{X}$ and Y be the solution of the SDEs:

$$\begin{aligned} d\mathbf{X}_r^{t,\mathbf{x},\alpha} &= B(tr\mathbf{X}_r^{t,\mathbf{x},\alpha},\alpha_r)dr + \Sigma(r,\mathbf{X}_r^{t,\mathbf{x},\alpha},\alpha_r)dW_r, \\ dY_r^{t,\mathbf{x}y,,\alpha} &= b(r,\mathbf{X}_r^{t,\mathbf{x},\alpha},Y_r^{t,\mathbf{x},y,\alpha},\alpha_r)dt + \Sigma(r,\mathbf{X}_r^{t,\mathbf{x},\alpha},Y_r^{t,\mathbf{x},y,\alpha},\alpha_r)dW_r, \end{aligned} \tag{A.1}$$

with $(\mathbf{X}_t^{t,\mathbf{x},\alpha},Y_t^{t,\mathbf{x}y,,\alpha}) = (\mathbf{x},y)$, where (B,Σ) and (b,σ) are respectively H-valued and $\mathbb{R}$-valued coefficients satisfying the usual regularity conditions, and α lives in a set of controls $\mathcal{A}$. We assume that the elements of $\mathcal{A}$ are càdlàg, progressively mesurable and take their values in a convex, open and unbounded Polish space $\mathbb{A}$.
We consider the following problem : given $g:H\to\mathbb{R}$ and $(t,\mathbf{x},y)\in[0,T]\times H\times\mathbb{R}$, we want to determine the following set of controls:

$$\mathcal{A}(t,\mathbf{x},y) := \{\alpha\in\mathcal{A} : Y_T^{t,\mathbf{x},y,\alpha}\geq g(\mathbf{X}_T^{t,\mathbf{x},y,\alpha})\}.$$

Similarly to [37], we observe that the monotonicity of Y in its initial condition y along with the independence of $\mathbf{X}$ from y provides the following result:

$$\mathcal{A}(t,\mathbf{x},y)\neq\emptyset\ \Rightarrow \mathcal{A}(t,\mathbf{x},y')\ \forall y'\geq t.$$

This leads us to define and study the following function:

$$w(t,\mathbf{x}) := \inf\{y\in\mathbb{R} = \mathcal{A}(t,\mathbf{x},y)\neq\emptyset\}. \tag{A.2}$$

As in [37], we shall prove that w is a viscosity solution to some second order partial differential equation (here on the Hilbert space H). Our first observation is that w satisfies the dynamic programming principle (DPP for short):

Proposition A.1. *For all $(t,\mathbf{x})\in[0,T]\times H$ and all $[t,T]$-valued stopping time θ, we have:*

$$w(t,\mathbf{x}) = \inf\{y\in\mathbb{R} : \exists\alpha\in\mathcal{A},\ Y_\theta^{t,\mathbf{x},y,\alpha}\geq w(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})\ \mathbb{P}-a.s.\}. \tag{A.3}$$

Proof. We observe that the proof of the DPP in [36, Theorem 3.1] applies to infinite dimensional state and control spaces. Indeed, it relies on the following assumptions:
• $\mathcal{A}$ is separable and stable by concatenation, which is the case since our control set is a Polish space of càdlàg paths; in particular, those assumptions allow to apply Jankov-Von Neumann theorem for measurable selection.
• the assumptions Z1-Z5 in [36] on the state process, which are easily satisfies here as $(\mathbf{X},Y)$ satisfies a standard SDE in a Hilbert space. ■

We can now derive the corresponding dynamic programming equation. First, for $(t, \mathbf{x}, y, \mathbf{p}) \in [0,T] \times H \times \mathbb{R} \times H$, we introduce:

$$\mathcal{N}(t, \mathbf{x}, \mathbf{p}) := \{a \in \mathbb{A} : \sigma(t, \mathbf{x}, y, a) - \langle \Sigma(t, \mathbf{x}, a), \mathbf{p} \rangle = 0\}.$$

Then, we consider the following equation on $[0,T] \times H$:

$$-\partial_t u(t, \mathbf{x}) + \sup_{a \in \mathcal{N}(t, \mathbf{x}, u(t, \mathbf{x}), D_{\mathbf{x}} u(t, \mathbf{x}))} \Big\{ b(t, \mathbf{x}, u(t, \mathbf{x}), a) - \mathcal{L}^a u(t, \mathbf{x}) \Big\} = 0, \quad u(T, \mathbf{x}) = g(x), \tag{A.4}$$

with

$$\mathcal{L}^a u(t, \mathbf{x}) := \langle B(t, \mathbf{x}, a), D_{\mathbf{x}} u(t, \mathbf{x}) \rangle + \frac{1}{2} \langle \Sigma(t, \mathbf{x}, a), D^2_{\mathbf{x}\mathbf{x}} u(t, \mathbf{x}) \Sigma(t, \mathbf{x}, a) \rangle.$$

Definition A.2 (Viscosity solutions)**.** *Let $u : [0,T] \times H \longrightarrow \mathbb{R}$ be continuous.*

(i) u is said to be a viscosity super-solution of (A.4) if $u(T, \cdot) \geq G$ and, for all $\varphi \in C^{1,2}([0,T] \times H)$ such that $u - \varphi$ has a local minimum in $(t, \mathbf{x})$, we have

$$-\partial_t u(t, \mathbf{x}) + \sup_{a \in \mathcal{N}(t, \mathbf{x}, u(t, \mathbf{x}), D_{\mathbf{x}} u(t, \mathbf{x}))} \Big\{ b(t, \mathbf{x}, u(t, \mathbf{x}), a) - \mathcal{L}^a u(t, \mathbf{x}) \Big\} \geq 0.$$

(ii) u is said to be a viscosity sub-solution of (A.4) if $u(T, \cdot) \leq G$ and, for all $\varphi \in C^{1,2}([0,T] \times H)$ such that $u - \varphi$ has a local maximum in $(t, \mathbf{x})$, we have

$$-\partial_t u(t, \mathbf{x}) + \sup_{a \in \mathcal{N}(t, \mathbf{x}, u(t, \mathbf{x}), D_{\mathbf{x}} u(t, \mathbf{x}))} \Big\{ b(t, \mathbf{x}, u(t, \mathbf{x}), a) - \mathcal{L}^a u(t, \mathbf{x}) \Big\} \leq 0.$$

(iii) u is said to be a viscosity solution of (A.4) if it is both a viscosity super-solution and viscosity sub-solution of (A.4).

Theorem A.3. *Assume that w is continuous, and that $\mathcal{N}$ is continuous in the following sense: if $a_0 \in \mathcal{N}(t_0, \mathbf{x}_0, y_0, \mathbf{p}_0)$, then there exists a mapping $\hat{a} : [0,T] \times H \times \mathbb{R} \times H \to \mathbb{A}$ such that*

$$\begin{cases} \hat{a}(t_0, \mathbf{x}_0, y_0, \mathbf{p}_0) = a_0 \\ \hat{a}(t, \mathbf{x}, y, \mathbf{p}) \in \mathcal{N}(t, \mathbf{x}, y, \mathbf{p}) \ \forall (t, \mathbf{x}, y, \mathbf{p}) \in [0,T] \times H \times \mathbb{R} \times H. \end{cases}$$

Then w is a viscosity solution of (A.4).

Proof. The argument is very similar to [37]. However, as we are in an infinite dimensional setting and that we do not require $\mathbb{A}$ to be compact, we detail the proof.
(i) We first show the supersolution property. Let φ be a test function; we may assume without loss of generality that $\varphi(t, \mathbf{x}) = w(t, \mathbf{x})$ and that the minimum in the tangency property is global. Let $\alpha \in \mathcal{A}(t, \mathbf{x}, w(t, \mathbf{x}))$ and introduce, for $\delta > 0$,

$$\theta_\delta := \inf\{s \geq t : (s, \mathbf{X}_s^{t,\mathbf{x},\alpha}) \notin [t, t+\delta) \times B_\delta(\mathbf{x})\}, \tag{A.5}$$

where $B_\delta(\mathbf{x})$ is the ball of radius δ and center $\mathbf{x}$ in H. As a consequence of the DPP (A.3), we have:

$$Y_{\theta_\delta}^{t,\mathbf{x},w(t,\mathbf{x}),\alpha} \geq w(\theta_\delta, \mathbf{X}_{\theta_\delta}^{t,\mathbf{x},w(t,\mathbf{x}),\alpha}) \geq \varphi(\theta_\delta, \mathbf{X}_{\theta_\delta}^{t,\mathbf{x},w(t,\mathbf{x}),\alpha}), \ \mathbb{P}-\text{a.s.}.$$

Therefore, applying Itô's formula between t and θ_δ, we obtain:

$$\begin{aligned} &\textstyle\int_t^{\theta_\delta} \Big(b(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}, Y_s^{t,\mathbf{x},w(t,\mathbf{x}),\alpha}, \alpha_s) - \partial_t \varphi(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}) - \mathcal{L}^{\alpha_s} \varphi(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}) \Big) ds \\ &\quad + \textstyle\int_t^{\theta_\delta} \Big(\sigma(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}, Y_s^{t,\mathbf{x},w(t,\mathbf{x}),\alpha}, \alpha_s) - \langle \Sigma(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}, \alpha_s), D_{\mathbf{x}} \varphi(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}) \rangle \Big) dW_s \geq 0, \end{aligned}$$

$\mathbb{P}$-a.s. For $n \in \mathbb{N}$, we now introduce the measure $\mathbb{P}^n$ defined by

$$\frac{\mathbb{P}^n}{\mathbb{P}} = \mathcal{E}\Big(-n \int_t^{T \wedge \theta_\delta} \big(\sigma(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}, Y_s^{t,\mathbf{x},w(t,\mathbf{x}),\alpha}, \alpha_s) - \Sigma(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}, \alpha_s) \big) dW_s \Big),$$

so that taking the expectation under $\mathbb{P}^n$ in the above inequality provides:

$$\mathbb{E}^{\mathbb{P}^n} \Big[\textstyle\int_t^{\theta_\delta} \Big(b(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}, Y_s^{t,\mathbf{x},w(t,\mathbf{x}),\alpha}, \alpha_s) - \partial_t \varphi(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}) - \mathcal{L}^{\alpha_s} \varphi(s, \mathbf{X}_s^{t,\mathbf{x},\alpha}) \Big) ds$$

$$-n\int_t^{\theta_\delta}\Big(\sigma(s,\mathbf{X}_s^{t,\mathbf{x},\alpha},Y_s^{t,\mathbf{x},w(t,\mathbf{x}),\alpha},\alpha_s)-\langle\Sigma(s,\mathbf{X}_s^{t,\mathbf{x},\alpha},\alpha_s),D_{\mathbf{x}}\varphi(s,\mathbf{X}_s^{t,\mathbf{x},\alpha})\rangle\Big)^2 ds\Big]\geq 0,$$

Then, by standard argument and the fact that this inequality must be true for all $n\in\mathbb{N}$, we obtain:

$$b(s,\mathbf{x},w(t,\mathbf{x}),\alpha_t)-\partial_t\varphi(s,\mathbf{x})-\mathcal{L}^{\alpha_t}\varphi(s,\mathbf{x})\geq 0,$$

with α_t such that

$$\big(\sigma(t,\mathbf{x},w(t,\mathbf{x}),\alpha_t)-\langle\Sigma(t,\mathbf{x},\alpha_t),D_{\mathbf{x}}\varphi(t,\mathbf{x})\rangle\big)^2=0,$$

which proves the supersolution property.

(ii) We now prove the subsolution property. Given a test function φ, we may assume without loss of generality that $\varphi(t,\mathbf{x})=w(t,\mathbf{x})$ and that

$$\inf_{(s,\tilde{\mathbf{x}})\in\partial_p B_\delta(t,\mathbf{x})}\varphi(s,\tilde{\mathbf{x}})-w(s,\tilde{\mathbf{x}})\geq\delta>0, \tag{A.6}$$

where $\partial_p B_\delta(t,\mathbf{x}):=\{t+\delta\}\times \mathrm{cl}(B_\delta(\mathbf{x}))\cup[t,t+\delta]\times\partial B_\delta(\mathbf{x})$ is the parabolic border of $B_\delta(t,\mathbf{x}):=[t,t+\delta)\times B_\delta(\mathbf{x})$. This can be achieved for example by adding a term in $|\tilde{\mathbf{x}}-\mathbf{x}|_H^4$ to the test function.
We shall prove the subsolution property by contradiction. Assume that

$$-\partial_t u(t,\mathbf{x})+\sup_{a\in\mathcal{N}(t,\mathbf{x},u(t,\mathbf{x}),D_{\mathbf{x}}u(t,\mathbf{x}))}\Big\{b(t,\mathbf{x},u(t,\mathbf{x}),a)-\mathcal{L}^a u(t,\mathbf{x})\Big\}>0. \tag{A.7}$$

By continuity of $\mathcal{N}$, δ can be chosen so that

$$-\partial_t u(t,\mathbf{x})+\Big\{b(t,\mathbf{x},u(t,\mathbf{x}),\hat{a}(s,\tilde{\mathbf{x}},\varphi(s,\tilde{\mathbf{x}}),D_{\mathbf{x}}f(s,\tilde{\mathbf{x}})))-\mathcal{L}^{a(s,\tilde{\mathbf{x}},\varphi(s,\tilde{\mathbf{x}}),D_{\mathbf{x}}f(s,\tilde{\mathbf{x}}))}u(t,\mathbf{x})\Big\}\geq\delta$$

for all $(s,\tilde{\mathbf{x}})\in B_\delta(t,\mathbf{x})$, with the mapping $\hat{a}$ as in the assumptions of the Theorem. Fix now $\eta>0$, and let $(\mathbf{X}^\eta,Y^\eta)$ be the solution of the SDEs (A.1) such that

$$\mathbf{X}_t^\eta=\mathbf{x},\ Y_t^\eta=w(t,\mathbf{x})-\eta,$$

and controlled by $\hat{\alpha}_s:=\hat{a}(s,\mathbf{X}_s^\eta,Y_s^\eta,D_{\mathbf{x}}\varphi(s,X_s^\eta))$. Let also θ_δ be as in (A.5). We have:

$$\begin{aligned}Y_{\theta_\delta}^\eta-w(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta) &= Y_{\theta_\delta}^\eta-\varphi(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta)+\varphi(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta)-w(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta)\\ &\geq Y_{\theta_\delta}^\eta-\varphi(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta)+\delta\end{aligned}$$

by (A.6). Introduce now the process $\hat{Y}_s^\eta:=\varphi(s,\mathbf{X}_s^\eta)-\eta$. As in [37], we observe that $\hat{Y}^\eta$ satisfies the same SDE as Y^η with a lower drift term, due to our hypothesis (A.7). Therefore, by stochastic comparison, we have $\hat{Y}^\eta\leq Y^\eta$. Coming back to the previous inequalities, we have:

$$\begin{aligned}Y_{\theta_\delta}^\eta-w(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta) &\geq Y_{\theta_\delta}^\eta-\hat{Y}_{\theta_d}^\eta+\hat{Y}_{\theta_d}^\eta-\varphi(\theta_\delta,\mathbf{X}_{\theta_\delta}^\eta)+\delta\\ &\geq -\eta+\delta.\end{aligned}$$

Thus, taking $\eta<\delta$, we obtain a contradiction of the DPP (A.3). Thus (A.7) is false and the viscosity subsolution property is satisfied. ■

References

[1] E. Abi Jaber and O. El Euch. Multifactor approximation of rough volatility models. *SIAM Journal on Financial Mathematics*, 10(2):309–349, 2019.

[2] E. Abi Jaber, E. Miller, and H. Pham. Linear–quadratic control for a class of stochastic Volterra equations: solvability and approximation. *The Annals of Applied Probability*, 31(5):2244–2274, 2021.

[3] N. Agram, B. Oksendal, and S. Yakhlef. New approach to optimal control of stochastic Volterra integral equations. *Stochastics: An International Journal of Probability and Stochastic Processes*, 91(6):873–894, 2019.

[4] Nacira Agram and Bernt Oksendal. Malliavin calculus and optimal control of stochastic Volterra equations. *Journal of Optimization Theory and Applications*, 167:1070–1094, 2015.

[5] Aurélien Alfonsi and Ahmed Kebaier. Approximation of stochastic Volterra equations with kernels of completely monotone type. *Mathematics of Computation*, 93(346):643–677, 2024.

[6] C. Bayer, P. Friz, and J. Gatheral. Pricing under rough volatility. *Quantitative Finance*, 16(6):887–904, 2016.

[7] Christian Bayer and Simon Breneis. Markovian approximations of stochastic Volterra equations with the fractional kernel. *Quantitative Finance*, 23(1):53–70, 2023.

[8] B. Bouchard, R. Élie, and C. Imbert. Optimal control under stochastic target constraints. *SIAM Journal on Control and Optimization*, 48(5):3501–3531, 2010.

[9] H. Brézis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer New York, NY, 2011.

[10] P. Briand and F. Confortola. BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces. *Stochastic Processes and their Applications*, 118(5):818–838, 2008.

[11] A. Cárdenas, S. Pulido, and R. Serrano. Existence of optimal controls for stochastic Volterra equations. *ArXiv preprint arXiv:2207.05169*, 2022.

[12] G. di Nunno and M. Giordano. Lifting of Volterra processes: optimal control in UMD Banach spaces. *ArXiv preprint arXiv:2306.14175*, 2023.

[13] B. Dupire. Functional Itô calculus. Technical Report 2009–04–FRONTIERS, Bloomberg portfolio research paper, 2009.

[14] I. Ekren, C. Keller, N. Touzi, and J. Zhang. On viscosity solutions of path dependent PDEs. *The Annals of Probability*, 42(1):204–236, 2014.

[15] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[16] G. Fabbri, F. Gozzi, and A. Święch. *Stochastic optimal control in infinite dimension*, volume 82 of *Probability theory and stochastic modelling.* Springer Cham, 2017.

[17] Jean-Pierre Fouque and Ruimeng Hu. Optimal portfolio under fractional stochastic environment. *Mathematical Finance*, 29(3):697–734, 2019.

[18] J. Gatheral, T. Jaisson, and M. Rosenbaum. Volatility is rough. *Quantitative Finance*, 18(6):933–949, 2018.

[19] L. Gawarecki and V. Mandrekar. *Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations.* Probability and its applications. Springer Berlin, Heidelberg, 2011.

[20] Y. Hamaguchi. Markovian lifting and asymptotic log-Harnack inequality for stochastic Volterra integral equations. *ArXiv preprint arXiv:2304.06683*, 2023.

[21] Yushi Hamaguchi. On the maximum principle for optimal control problems of stochastic Volterra integral equations with delay. *Applied Mathematics & Optimization*, 87(3):42, 2023.

[22] P. Harms. Strong convergence rates for Markovian representations of fractional processes. *Discrete & Continuous Dynamical Systems–B*, 26(10):5567, 2021.

[23] C. Hernández and D. Possamaï. Me, myself and I: a general theory of non-Markovian time-inconsistent stochastic control for sophisticated agents. *The Annals of Applied Probability*, 33(2):1396–1458, 2023.

[24] C. Hernández and D. Possamaï. Time-inconsistent contract theory. *ArXiv preprint arXiv:2303.01601*, 2023.

[25] Camilo Hernández, Nicolás Hernández Santibáñez, Emma Hubert, and Dylan Possamaï. Closed-loop equilibria for Stackelberg games: it's all about stochastic targets. *Preprint arXiv:2406.19607*, 2024.

[26] I. Ito. On the existence and uniqueness of solutions of stochastic integral equations of the Volterra type. *Kodai Mathematical Journal*, 2(2):158–170, 1979.

[27] Ping Lin and Jiongmin Yong. Controlled singular Volterra integral equations and Pontryagin maximum principle. *SIAM Journal on Control and Optimization*, 58(1):136–164, 2020.

[28] P.-L. Lions. Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part I: the case of bounded stochastic evolutions. *Acta Mathematica*, 161(1):243–278, 1988.

[29] P.-L. Lions. Viscosity solutions of fully nonlinear second order equations and optimal stochastic control in infinite dimensions. Part II: optimal control of Zakai's equation. In G. da Prato and L. Tubaro, editors, *Stochastic partial differential equations and applications II. Proceedings of a conference held in Trento, Italy February 1–6, 1988*, volume 1390 of *Lecture notes in mathematics*, pages 147–170. Springer, 1989.

[30] P.-L. Lions. Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. III. Uniqueness of viscosity solutions for general second-order equations. *Journal of Functional Analysis*, 86(1):1–18, 1989.

[31] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *System and Control Letters*, 14(1):55–61, 1990.

[32] É. Pardoux and P.E. Protter. Stochastic Volterra equations with anticipating coefficients. *The Annals of Probability*, 18(4):1635–1655, 1990.

[33] P.E. Protter. Volterra equations driven by semimartingales. *The Annals of Probability*, 13(2):519–530, 1985.

[34] M. Saeedian, M. Khalighi, N. Azimi-Tafreshi, G.R. Jafari, and M. Ausloos. Memory effects on epidemic evolution: The susceptible-infected-recovered epidemic model. *Physical Review E*, 95(2):022409, 2017.

[35] J. Schmiegel. Self-scaling tumor growth. *Physica A: Statistical Mechanics and its Applications*, 367:509–524, 2006.

[36] H.M. Soner and N. Touzi. Dynamic programming for stochastic target problems and geometric flows. *Journal of the European Mathematical Society*, 4(3):201–236, 2002.

[37] H.M. Soner and N. Touzi. Stochastic target problems, dynamic programming, and viscosity solutions. *SIAM Journal on Control and Optimization*, 41(2):404–424, 2002.

[38] F. Viens and J. Zhang. A martingale approach for fractional Brownian motions and related path dependent PDEs. *The Annals of Applied Probability*, 29(6):3489–3540, 2019.

[39] H. Wang, J. Yong, and J. Zhang. Path dependent Feynman–Kac formula for forward backward stochastic Volterra integral equations. *Annales de l'institut Henri Poincaré, Probabilités et Statistiques* (B), 58(2):603–638, 2022.

[40] H. Wang, J. Yong, and C. Zhou. Linear–quadratic optimal controls for stochastic Volterra integral equations: causal state feedback and path-dependent Riccati equations. *SIAM Journal on Control and Optimization*, 61(4):2595–2629, 2023.

[41] J. Zhang. *Backward stochastic differential equations—from linear to fully nonlinear theory*, volume 86 of *Probability theory and stochastic modelling*. Springer-Verlag New York, 2017.